ISRAEL JOURNAL OF MATHEMATICS, Vol. 72, Nos. 1-2, 1990

COMPUTING THE BRAUER-LONG GROUP OF A HOPF ALGEBRA I: THE COHOMOLOGICAL THEORY

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ABSTRACT

Let H be a commutative, cocommutative and faithfully projective Hopf algebra over a commutative ring R. Using cohomological methods, we obtain a description of a subgroup of Long's Brauer group of H-dimodule algebras:

$$\mathrm{BD}^s(R,H) \cong \mathrm{Gal}^s(R,H) \times \mathrm{Gal}^s(R,H^*) \times_{\psi} \mathrm{Br}(R),$$

where the map ψ is induced by the smash product, and where $\mathrm{BD}^s(R,H)$ is the subgroup of the Brauer–Long group consisting of all elements which are split by a faithfully flat extension of R. As an example, the Brauer–Long group of a free Hopf algebra of rank 2 is computed. The results are also applied to Orzech's subgroup of the Brauer–Long group.

0. Introduction

Between 1969 and 1975, several generalizations of the Brauer–Wall group of a commutative ring have been proposed by different authors. The most natural of these generalizations was proposed by Long in [25]: he defined a group classifying equivalence classes of algebras equipped with an H-dimodule structure, H being a commutative, cocommutative and faithfully projective Hopf algebra over a commutative ring R. All previously introduced generalizations turned out to be special cases and/or subgroups of Long's Brauer group.

Long's Brauer group is usually denoted by $\mathrm{BD}(R,H)$, and has been studied

Received April 5, 1990

mainly in the case where H is a group ring over a finite abelian group G; we refer to [6–8, 10, 13, 15, 16, 24, 30, 31, 36], for different computations. The Brauer–Long group has a complicated arithmetic structure, and is connected to other interesting invariants of R and G, such as the Brauer group, the group of Galois extensions and the automorphism group of G.

So far, almost no specific results have been obtained in the case where H is not a group ring. M. Beattie pointed out its relation to the group of H-Galois objects (cf. [4, 5]), but, apart from that, little is known. In this paper, we generalize the methods applied by S. Caenepeel and M. Beattie to the case of a Hopf algebra. This is done in two steps: first, one considers the subgroup $BD^s(R, H)$ of $BD(R, H)$ consisting of all elements split by a faithfully flat R-algebra S. This subgroup may be described using derived functor cohomology on the flat site; it is shown in Section 3 that $BD^s(R, H)$ is a twisted product of the first cohomology groups of the sheaves of grouplike elements of H and its dual, and of the torsion part of the second cohomology group of the multiplicative group of units. Equivalently, one may state that $BD^s(R, H)$ is a twisted product of the groups of commutative H-Galois and H^*-Galois objects, and of the Brauer group.

The second step will be to compute the quotient $BD(R, H)/BD^s(R, H)$; this will be the topic of the forthcoming paper [9]. It is expected that $BD(R, H)/BD^s(R, H)$ is a well-defined subgroup of the group of Hopf algebra automorphisms of $H \otimes H^*$. The methods involved will be mainly based on the Skolem–Noether Theorem and its generalizations.

After introducing some notations and stating some preliminary results in Section 1, we give a cohomological description of the group of commutative H-Galois objects; it is shown that this group is isomorphic to the first cohomology group of the sheaf of grouplike elements of the dual Hopf algebra. As an application, we give a cohomological proof for the Kummer exact sequence in the case of a Hopf algebra. In Section 3, we describe $BD^s(R, H)$ as the twisted product of three cohomology groups. Here we generalize Villamayor and Zelinsky's construction of the embedding $Br(R) \to H^2(R, \mathbb{G}_m)$ (cf. [38]). A particular example is given in Section 4, where we discuss the Brauer–Long group of a free Hopf algebra of rank 2. For instance, if $R = \mathbb{Z}[\sqrt{2}]$, and H is the self-dual Hopf algebra of rank 2, then $BD(R, H) \cong D_8$, the dihedral group of order 16. In a final section, our results are applied to Orzech's subgroup of the Brauer–Long group (cf. [31]).

1. Notations and preliminary results

1.1. *Modules, comodules and dimodules*

Throughout this paper, R will be a commutative ring with unit, and H will be a Hopf algebra, which is, unless stated otherwise explicitly, commutative, cocommutative and faithfully projective as an R-module. We call an R-module faithfully projective if it is finitely generated, faithfully flat and projective. We use the following notations for the structural maps of H:

$\Delta = \Delta_H : H \to H \otimes H$, the diagonal map;

$\varepsilon = \varepsilon_H : H \to R$, the counit map;

$\mu = \mu_H : H \otimes H \to H$, the multiplication map;

$\eta = \eta_H : R \to H$, the unit map;

$S = S_H : H \to H$, the antipode.

We will often omit the subscript H. We will also make extensive use of Sweedler's notation; for example, we will write, for $h \in H$,

$$\Delta h = \sum_{(h)} h_{(1)} \otimes h_{(2)},$$

$$(1 \otimes \Delta)\Delta h = \sum_{(h)} h_{(1)} \otimes h_{(2)} \otimes h_{(3)},$$

etc. For more details concerning the theory of Hopf algebras, we refer to [1, 34]. $G(H) = \{h \in H : \Delta h = h \otimes h \text{ and } \Sigma(h) = 1\}$ is the group of group-like elements of H, and will play a key rôle in the sequel. Also recall that H^* is also a Hopf algebra, and that

$$\Delta_{H^*} = (\mu_H)^*, \quad \mu_{H^*} = (\Delta_H)^*, \quad \varepsilon_{H^*} = (\eta_H)^*, \quad \eta_{H^*} = (\mu_H)^*, \quad S_{H^*} = (S_H)^*.$$

A (left) H-module M is an R-module M, together with a map

$$\psi = \psi_M : H \otimes M \to M$$

satisfying

(1) $\psi(hg \otimes m) = \psi(h \otimes \psi(g \otimes m))$,

(2) $\psi(\eta(r) \otimes m) = rm$,

for all $h, g \in H$, $m \in M$, $r \in R$. In the sequel, we will often write

$$\psi(h \otimes m) = h \rightharpoonup m.$$

Also write

$$M^H = \{m \in M : h \rightharpoonup m = \varepsilon(h)m, \text{ for all } h \in H\}.$$

An R-module homomorphism f between two H-modules M, N is called an H-module homomorphism if it preserves the H-module structure, that is

$$\psi_N \circ (1 \otimes f) = f \circ \psi_M.$$

H-mod will denote the category of H-modules and H-module homomorphisms. If M and N are two H-modules, we can define an H-module structure on $M \otimes N$ and $\mathrm{Hom}(M, N)$ as follows: ($h \in H$, $m \in M$, $n \in N$, $f: M \to N$)

$$h \rightharpoonup (m \otimes n) = \sum_{(h)} (h_{(1)} \rightharpoonup m) \otimes (h_{(2)} \rightharpoonup n),$$

$$(h \rightharpoonup f)(m) = \sum_{(h)} h_{(1)} \rightharpoonup (f(S(h_{(2)}) \rightharpoonup m)).$$

In a dual way, a (right) H-comodule is an R-module M, together with a map

$$\chi = \chi_H : M \to M \otimes H$$

satisfying

(1) $(\chi \otimes 1)\chi = (1 \otimes \Delta)\chi$,

(2) $(1 \otimes \varepsilon)\chi = 1$.

We will also use Sweedler's notation for comodules:

$$\chi(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}.$$

A homomorphism $f: M \to N$ is called an H-comodule homomorphism if

$$\chi_N \circ f = (f \otimes 1) \circ \chi_M.$$

Com-H will be the category of (right) H-comodules and H-comodule homomorphisms.

We may define a functor

$$F: \text{com-}H \to H^*\text{-mod}$$

as follows: on $M \in \text{com-}H$, we define an H^*-module structure by letting

$$h^* \rightharpoonup m = \sum_{(m)} h^*(m_{(1)})m_{(0)}$$

for all $m \in M$, $h^* \in H^*$. Using the fact that H is faithfully projective, we may show that F is an equivalence of categories. Using this equivalence, we may define an H-comodule structure on $\mathrm{Hom}(M, N)$ and $M \otimes N$, for H-comodules M and N. This structure is given by the following formulas:

$$\chi_{M\otimes N}(m \otimes n) = \sum_{(m),(n)} m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)},$$

$$\chi(f)(m \otimes 1) = \sum_{(m),(f(m_{(0)}))} f(m_{(0)})_{(0)} \otimes f(m_{(0)})_{(1)}S(m_{(1)}).$$

An R-module M which is at once an H-module and an H-comodule is called an H-dimodule if for all $m \in M$, $h \in H$,

$$\sum_{(h\to m)} (h \to m)_{(0)} \otimes (h \to m)_{(1)} = \sum_{(m)} h \to m_{(0)} \otimes m_{(1)},$$

that is

$$\chi \circ \psi = (\psi \otimes 1) \circ \chi.$$

H-dim is the category of H-dimodules and H-dimodule homomorphisms. An H-dimodule homomorphism is an R-homomorphism which is an H-module and an H-comodule homomorphism. Applying the functor F defined above, and its inverse equivalent, we obtain equivalences of categories

$$\text{com-}(H \otimes H^*) \cong H\text{-dim} \cong (H^* \otimes H)\text{-mod}.$$

1.2. *H-module, H-comodule and H-dimodule algebras*

Let A be an H-module and an R-algebra. We say that A is an H-module algebra if for all $h \in H$ and $a, b \in A$:

(1) $h \to 1 = \varepsilon(h)1$,

(2) $h \to (ab) = \Sigma_{(h)} (h_{(1)} \to a)(h_{(2)} \to b)$.

Similarly, if A is an H-comodule and an R-algebra, then we call A an H-comodule algebra if for all $h \in H$ and $a, b \in A$:

(1) $\chi(1) = 1 \otimes 1$,

(2) $\chi(ab) = \chi(a)\chi(b)$.

If A is an H-dimodule, an H-module algebra and an H-comodule algebra, then we call A an H-dimodule algebra.

Given an H-module algebra A and an H-comodule algebra B, we define the smash product $A \# B$ of A and B as the tensor product $A \otimes B$ of an R-module and with multiplication defined by

$$(a \# b)(c \# d) = \sum_{(b)} a(b_{(1)} \to c) \# b_{(0)}d.$$

If A and B are dimodule algebras, then the smash product $A\#B$ furnished with the dimodule structure on $A \otimes B$ is again a dimodule algebra.

The H-opposite algebra $\bar{A}$ of an H-dimodule algebra A is equal to A as an H-dimodule, but with multiplication structure given by

$$\bar{a} \cdot \bar{b} = \overline{\sum_{(a)} (a_{(1)} \rightarrow b) a_{(0)}}.$$

1.3. *Some notations*

In the sequel S will always be a commutative faithfully flat R-algebra. $S^{(n)}$ will be a shorter notation for the tensor product of n copies of S over R. The tensor product over $S^{(n)}$ will be denoted by $\otimes_n$, and the tensor product over R will be simply denoted by $\otimes$. We also define maps

$$\varepsilon_i : S^{(n)} \rightarrow S^{(n+1)}$$

by

$$\varepsilon_i(s_1 \otimes \cdots \otimes s_n) = s_1 \otimes \cdots \otimes 1 \otimes \cdots \otimes s_n$$

for $i = 1, \ldots, n$. If $M \in S^{(n)}$-mod, then we define

$$M_i = M \otimes_n S^{(n+1)},$$

where $\varepsilon_i : S^{(n)} \rightarrow S^{(n+1)}$ is an $S^{(n)}$-module.

1.4. *Lemma (Faithfully flat descent for H-comodules and H-comodule algebras)*

Let S be a faithfully flat commutative R-algebra, and let M be an $H \otimes S$-comodule. Suppose that $g : S \otimes M \rightarrow M \otimes S$ is a descent datum in the sense of [21], *and that g is an $H \otimes S^{(2)}$-comodule isomorphism. Then there exists an H-comodule N and an $H \otimes S$-comodule isomorphism $\omega : N \otimes S \rightarrow M$ such that we have the following commutative diagram of $H \otimes S^{(2)}$-comodule isomorphisms:*

$$\begin{array}{ccc} S \otimes N \otimes S & \xrightarrow{1 \otimes \omega} & S \otimes M \\ \downarrow \tau \otimes 1 & & \downarrow g \\ N \otimes S \otimes S & \xrightarrow[\omega \otimes 1]{} & M \otimes S \end{array} \qquad (1.4.1)$$

(N, ω) is unique up to H-comodule isomorphism; furthermore, if M has the

structure of $H \otimes S$-comodule algebra, and if g is an $H \otimes S^{(2)}$-comodule algebra isomorphism, then N has a unique H-comodule structure such that ω is an $H \otimes S$-comodule algebra isomorphism.

Similar properties hold for H-modules and H-module algebras, and for H-dimodules and H-dimodule algebras.

PROOF. Let $N = \{x \in M : x \otimes 1 = g(1 \otimes x)\}$. It is well-known (cf. [21]) that N is an R-module, and that $\omega : N \otimes S \to M$, given by $\omega(n \otimes s) = ns$, is an S-module homomorphism making (1.4.1) commutative. Therefore, we only have to show that the comodule structure map $\chi_M : M \to M \otimes H \otimes S$ restricts to a map $\chi : N \to N \otimes H$. Observe that the map

$$g \otimes 1 : (S \otimes M) \otimes_2 (S \otimes H \otimes S) \cong S \otimes (M \otimes H \otimes S)$$

$$\to (M \otimes S) \otimes_2 (H \otimes S \otimes S) \cong (M \otimes H \otimes S) \otimes S$$

is a descent datum for $M \otimes H \otimes S$, and that the descended module is $N \otimes H$. Therefore, it suffices to verify that for all $n \in N$,

$$\chi_M(n) \otimes 1 = (g \otimes 1)(1 \otimes \chi_M(n)).$$

Invoking the fact that g is an $H \otimes S^{(2)}$-comodule isomorphism, we have:

$$\begin{aligned} \chi_M(n) \otimes 1 &= \chi_{M \otimes S}(n \otimes 1) \\ &= \chi_{M \otimes S}(g(1 \otimes n)) \\ &= (g \otimes 1)(\chi_{M \otimes S}(1 \otimes n)) \\ &= (g \otimes 1)(1 \otimes \chi_M(n)). \end{aligned}$$

The uniqueness follows easily: suppose that (N, ω) and (K, κ) are two solutions of the problem. Then $\rho = \omega^{-1}\kappa : K \otimes S \to N \otimes S$ is an isomorphism of $H \otimes S$-comodules. In [21], Knus and Ojanguren show that $\rho = 1 \otimes \alpha$, for some $\alpha : K \to N$. Since ρ is an $H \otimes S$-comodule isomorphism, it follows that α is an H-comodule isomorphism.

If M is an $H \otimes S$-comodule algebra, and g is an $H \otimes S^{(2)}$-comodule algebra isomorphism, then, according to [21], N may be furnished with an algebra structure. Arguments similar to the one given above show that N is an H-comodule algebra.

The corresponding properties for H-modules and H-dimodules follow by duality: one uses the category equivalences H-mod $\cong$ com-H and com-$(H \otimes H^*) = H$-dim.

1.5. *Invertible H-dimodules*

Consider the following categories:

Category	Objects	Morphisms
FPC(R, H)	faithfully projective H-comodules	H-comodule homomorphisms
FPM(R, H)	faithfully projective H-modules	H-module homomorphisms
FPD(R, H)	faithfully projective H-dimodules	H-dimodule homomorphisms
PC(R, H)	invertible H-comodules	H-comodule homomorphisms
PM(R, H)	invertible H-modules	H-module homomorphisms
PD(R, H)	invertible H-dimodules	H-dimodule homomorphisms

Let $K_0\mathbf{PC}(R, H) = \mathrm{PC}(R, H)$, $K_0\mathbf{PM}(R, H) = \mathrm{PM}(R, H)$, $K_0\mathbf{PD}(R, H) = \mathrm{PD}(R, H)$. These new versions of the Picard group may be easily related to the classical Picard group $\mathrm{Pic}(R)$.

1.6. *Proposition*

(1.6.1) $\mathrm{PC}(R, H) \cong \mathrm{Pic}(R) \times G(H)$;
(1.6.2) $\mathrm{PM}(R, H) \cong \mathrm{Pic}(R) \times G(H^*)$;
(1.6.3) $\mathrm{PD}(R, H) \cong \mathrm{Pic}(R) \times G(H) \times G(H^*)$.

PROOF.
(1.6.1) We define a map

$$F : \mathrm{Pic}(R) \times G(H) \to \mathrm{PC}(R, H)$$

as follows: $F([I], h) = [I(h)]$, where $I(h)$ is equal to R as an R-module, and with H-comodule structure given by

$$\chi(x) = x \otimes h$$

for all $x \in I$. This defines a comodule structure, since, for all $x \in H$:

$$(\chi \otimes 1)(\chi(x)) = (\chi \otimes 1)(x \otimes h) = x \otimes h \otimes h = (1 \otimes \Delta)(\chi(x))$$

and

$$(1 \otimes \varepsilon)(\chi(x)) = (1 \otimes \varepsilon)(x \otimes h) = \varepsilon(h)x = x.$$

Let us show that F is an isomorphism of groups; it is clear that F is a well-defined homomorphism.

Consider $I \in \mathbf{PC}(R, H)$, and let $\underline{I}$ be equal to I as an R-module. Furnish $\underline{I}$

with the trivial H-comodule structure. Then $\underline{I}^*\otimes I=J\cong R$ as an R-module. Therefore

$$\chi_J(1)=1\otimes h$$

for some $h\in H$. Then $h\in G(H)$. Indeed,

$$1=(1\otimes\varepsilon)\chi(1)=(1\otimes\varepsilon)(1\otimes h)=\varepsilon(h)$$

and

$$1\otimes\Delta h=(1\otimes\Delta)\chi(1)=(\chi\otimes 1)\chi(1)=(\chi\otimes 1)(1\otimes h)=1\otimes h\otimes h.$$

Define $G:\mathrm{PC}(R,H)\to\mathrm{Pic}(R)\times G(H)$ by

$$G([I])=([\underline{I}],h).$$

It is straightforward to show that F and G are each others inverses, proving (1.6.1).

(1.6.2) and (1.6.3) follow from the equivalences of the categories

$$\mathbf{PM}(R,H)\quad\text{and}\quad\mathbf{PC}(R,H^*)$$

and

$$\mathbf{PD}(R,H)\quad\text{and}\quad\mathbf{PC}(R,H\otimes H^*),$$

and after applying Lemma 1.7 below. For later use, we give an explicit description of the isomorphism

$$F:\mathrm{Pic}(R)\times G(H^*)\to\mathrm{PM}(R,H).$$

Let $F([I],h^*)=[I(h^*)]$, where $I(h^*)$ is the R-module I with H-module structure defined by

$$h\rightharpoonup x=h^*(h)x$$

for all $h\in H$, $x\in I$. Similarly, for $(h,h^*)\in G(H)\times G(H^*)\cong G(H\otimes H^*)$, we define $I(h,h^*)$ by

$$I=I(h,h^*)\quad\text{as } R\text{-modules},$$

$$\chi(x)=x\otimes h,\quad\text{for all } x\in I,$$

$$k\rightharpoonup x=h^*(k)x,\quad\text{for all } k\in H,\quad x\in I.$$

In the proof of Proposition 1.6, we implicitly used the following Lemma:

1.7. *Lemma*

Let H, K be Hopf algebras. Then $G(H)\times G(K)\cong G(H\otimes K)$.

Proof. It is clear that we have an embedding $G(H)\times G(K)\to G(H\otimes K)$, by mapping (h,k) to $h\otimes k$. Let us show that this embedding is onto. Take $\Sigma_i h_i\otimes k_i\in G(H\otimes K)$. Then

$$h=\sum_i \varepsilon(k_i)h_i\in G(H) \quad\text{and}\quad k=\sum_i \varepsilon(h_i)k_i\in G(K).$$

From the fact that $\Sigma_i h_i\otimes k_i$ is grouplike, it follows that

$$\sum_{i,(h_i),(k_i)} h_{i_{(1)}}\otimes k_{i_{(1)}}\otimes h_{i_{(2)}}\otimes k_{i_{(2)}}=\sum_{i,j} h_i\otimes k_i\otimes h_j\otimes k_j.$$

Applying $1\otimes\varepsilon\otimes\varepsilon\otimes 1$ to both sides, we obtain

$$\sum_i h_i\otimes k_i=h\otimes k.$$

1.8. *Proposition*

Let $P,Q\in\mathbf{FPD}(R,H)$, and suppose that $\Phi:\mathrm{End}(P)\to\mathrm{End}(Q)$ is an H-dimodule algebra isomorphism. Then there exists $I\in\mathbf{PD}(R,H)$ and an H-dimodule isomorphism $f: P\otimes I\to Q$ such that f induces Φ, that is

$$\Phi(\rho)=f\circ(\rho\otimes 1)\circ f^{-1}$$

for all $\rho\in\mathrm{End}(P)$. Furthermore, f and I are unique up to dimodule isomorphism: $f: P\otimes I\to Q$ and $g: P\otimes J\to Q$ induce the same isomorphism Φ if and only if there exists an H-dimodule isomorphism $u: I\to J$ such that $f=g\circ(1\otimes u)$.

Proof. Recall from [4, Ch. 3, ex. 3.2] that

$$(A=\mathrm{End}(P), R, P, P^*, \varphi, \psi)$$

with

$$\varphi: P\otimes P^*\to A \quad\text{and}\quad \psi: P^*\otimes_A P\to R$$

defined by

$$\varphi(p\otimes p^*)(x)=p^*(x)p \quad\text{and}\quad \psi(p^*\otimes p)=p^*(p)$$

is a so-called strict Morita H-context. As a consequence (cf. [4, Ch. 3, Prop. 3.3]), the categories $(H, A\text{-}R)$-mod and R-dim are equivalent. Here

$(H, A\text{-}R)$-mod is the category of $(H, A\text{-}R)$-bimodules. An $(H, A\text{-}R)$-bimodule is an A-R-bimodule M which is also an H-dimodule, such that the following properties hold for all $h \in H$, $m \in M$, $a \in A$:

(1) $h \rightarrow (am) = \Sigma_{(h)} (h_{(1)} \rightarrow a)(h_{(2)} \rightarrow m)$;

(2) $\chi_M(am) = \Sigma_{(a),(m)} a_{(0)}m_{(0)} \otimes a_{(1)}m_{(1)}$.

The inverse equivalences between those categories are given by

$$P^* \otimes_A . : (H, A\text{-}R)\text{-mod} \rightarrow R\text{-dim};$$

$$P \otimes . : R\text{-dim} \rightarrow (H, A\text{-}R)\text{-mod}.$$

Now Q is an object of the category $(H, A\text{-}R)$-mod if we give Q a left A-action as follows:

$$\rho \,.\, q = \Phi(\rho)(q)$$

for all $\rho \in A = \mathrm{End}(P)$ and $q \in Q$.

Indeed, for all $h \in H$, $\rho \in A$ and $q \in Q$ we have:

$$\begin{aligned}\sum_{(h)} (h_{(1)} \rightarrow \rho) \,.\, (h_{(2)} \rightarrow q) &= \sum_{(h)} \Phi(h_{(1)} \rightarrow \rho)(h_{(2)} \rightarrow q)\\ &= \sum_{(h)} (h_{(1)} \rightarrow \Phi(\rho))(h_{(2)} \rightarrow q)\\ &= \sum_{(h)} h_{(1)} \rightarrow (\Phi(\rho)(S(h_{(2)} \rightarrow (h_{(3)} \rightarrow q))))\\ &= \sum_{(h)} h_{(1)} \rightarrow \Phi(\rho)(\varepsilon(h_{(2)})q)\\ &= h \rightarrow \Phi(\rho)(q)\\ &= h \rightarrow (\rho \,.\, q).\end{aligned}$$

The second condition for being an $(H, A\text{-}R)$-bimodule holds by a duality argument. Therefore $Q \cong (P^* \otimes_A Q) \otimes P \cong P \otimes (P^* \otimes_A Q)$ as H-dimodules. By a rank argument, the rank of I has to be 1, so I is invertible, and is therefore an object of $\mathbf{PD}(R, H)$. Let $f: P \otimes I \rightarrow Q$ be the involved isomorphism. Then for all $q \in Q$, $\rho \in \mathrm{End}(P)$, we have

$$\begin{aligned}\Phi(\rho)(q) &= \rho \,.\, q\\ &= (f \circ f^{-1})(\rho \,.\, q)\\ &= f(\rho \otimes 1) f^{-1}(q).\end{aligned}$$

Hence $\Phi(\rho) = f \circ (\rho \otimes 1) \circ f^{-1}$.

1.9. *Cohomology on the flat site*

Let R_{fl} be the following category:

objects: $R \to S$, where S is a flat R-algebra;
morphisms: commutative diagrams of the form

$$\begin{array}{ccc} R & \to & S \\ & \searrow & \downarrow \\ & & T \end{array}$$

where T is a flat S-algebra.

$\mathbf{P}(R_{\mathrm{fl}})$ is the category of covariant functors (presheaves) from R_{fl} to abelian groups. An object $P \in \mathbf{P}(R_{\mathrm{fl}})$ is called a sheaf if for any morphism $S \to S'$ in R_{fl}, such that $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$ is surjective, the sequence

$$1 \to P(S) \to P(S') \rightrightarrows P(S' \otimes_S S')$$

is exact; here ε_1 and $\varepsilon_2 : S' \to S' \otimes_S S'$ are given by $\varepsilon_1(s) = 1 \otimes s$, $\varepsilon_2(s) = s \otimes 1$, and applying the functor P, we obtain maps $P(\varepsilon_1), P(\varepsilon_2) : P(S') \to P(S' \otimes_S S')$.

$\mathbf{S}(R_{\mathrm{fl}})$ is the full category of $\mathbf{P}(R_{\mathrm{fl}})$, with sheaves as objects. It may be shown that $\mathbf{P}(R_{\mathrm{fl}})$ and $\mathbf{S}(R_{\mathrm{fl}})$ are Grothendieck categories having enough injective objects.

Consider the global section function $\Gamma : \mathbf{P}(R_{\mathrm{fl}}) \to \mathbf{Ab}$, defined by $\Gamma(F) = F(R)$. The n-th flat cohomology group of a sheaf F is defined by $H^n(R_{\mathrm{fl}}, F) = R^n\Gamma(F)$, the n-th right derived functor of Γ, evaluated at F.

In a similar way, we may define the étale cohomology groups $H^n(R_{\mathrm{ét}}, F)$ (replacing all flat morphisms used above by étale morphisms). All the cohomology groups used in this paper will be flat cohomology groups, and therefore we will omit the subscript "fl".

Let S be a faithfully flat R-algebra, and $H^n(S/R, F)$ the n-th Amitsur cohomology group of F (cf. [21], for example, for a definition of Amitsur cohomology). It is well-known that we have an embedding

$$1 \to H^n(S/R, F) \to H^n(R, F).$$

For $n = 1$, $H^1(R, F)$ is fully described by Amitsur cohomology:

$$H^1(R, F) \cong \varinjlim H^1(S/R, F),$$

where the inductive limit is taken over all faithfully flat R-algebras S. It follows from Artin's Refinement Theorem ([2]) that the statement above also holds for $n > 1$, if we take the cohomology on the ètale site.

It is well-known that $\mathbb{G}_m(S) = \{x \in S : x \text{ is invertible in } S\}$ is a sheaf on R_{fl}. The following sheaf will also be important in the sequel:

1.10. *Proposition*

Let H be a commutative, cocommutative and faithfully projective Hopf algebra. Then $G(H \otimes .)$ is a sheaf on R_{fl}.

PROOF. Let $f: S \to S'$ be a faithfully flat S-algebra. We have to show that

$$\begin{aligned} 1 \to G(S \otimes H) &\to G(S' \otimes H) \\ &\rightrightarrows G(S' \otimes_S S') \otimes H) \\ &= G((S' \otimes H) \otimes_{S \otimes H} (S' \otimes H)) \end{aligned}$$

is exact. We have a commutative diagram

$$\begin{array}{ccccc} & 1 & & 1 & & 1 \\ & \downarrow & & \downarrow & & \downarrow \\ 1 \to & G(S \otimes H) & \to & G(S' \otimes H) & \rightrightarrows & G((S' \otimes H) \otimes_{S \otimes H} (S' \otimes H)) \\ & \downarrow & & \downarrow & & \downarrow \\ 1 \to & \mathbb{G}_m(S \otimes H) & \to & \mathbb{G}_m(S' \otimes H) & \rightrightarrows & \mathbb{G}_m((S' \otimes H) \otimes_{S \otimes H} (S' \otimes H)) \end{array}$$

The bottom row is exact since $\mathbb{G}_m$ is a sheaf. So we have to show only one thing: does $(f \otimes 1)(x) \in G(S' \otimes H)$ imply that $x \in G(S \otimes H)$? This may be seen as follows:

$$\begin{aligned} ((f \otimes 1) \otimes (f \otimes 1))(x \otimes x) &= (f \otimes 1)(x) \otimes (f \otimes 1)(x) \\ &= \Delta_{S'}(f \otimes 1)(x) \\ &= ((f \otimes 1) \otimes (f \otimes 1))\Delta_S(x). \end{aligned}$$

hence $\Delta_S(x) = x \otimes x$, since

$$(f \otimes 1) \otimes (f \otimes 1) : (S \otimes H) \otimes_S (S \otimes H) \to (S' \otimes H) \otimes_{S'} (S' \otimes H)$$

is faithfully flat.

2. A cohomological description of the group of commutative H-Galois objects

Throughout this section, H is a faithfully projective commutative and cocommutative Hopf algebra over a commutative ring R. Recall the following definition from [12]:

2.1. *Definition*

An H-comodule algebra A is called an H-Galois object if the following two conditions hold:

(2.1.1) $\gamma : A \otimes \underline{A} \to H \otimes \underline{A}$, defined by $\gamma(x \otimes y) = \Sigma_{(x)} x_{(1)} \otimes x_{(0)} y$ is an isomorphism of H-comodule algebras;

(2.1.2) $\underline{A}$ is faithfully flat as an R-module.

Here $\underline{A}$ denotes A without its comodule structure, that is $\_$ is the functor forgetting the comodule structure. The set of H-comodule algebra isomorphism classes of commutative H-Galois objects will be denoted by $\mathrm{Gal}^s(R, H)$. It is known that $\mathrm{Gal}^s(R, H)$ forms an abelian group under the following operation:

$$A \,.\, B = \{x \in A \otimes B : \tau_1(\chi_S \otimes 1)(x) = (1 \otimes \chi_T)(x)\}$$
$$= \left\{ \sum_i a_i \otimes b_i \in A \otimes B : \sum_i a_{i_{(0)}} \otimes b_i \otimes a_{i_{(1)}} = \sum_i a_i \otimes b_{i_{(0)}} \otimes b_{i_{(1)}} \right\}.$$

Here $\tau_1 : A \otimes H \otimes B \to A \otimes B \otimes H$ is the switch map between the second and the third factor. The comodule map $\chi_{A\,.\,B}$ is given by $\chi_{A\,.\,B} = 1 \otimes \chi_B = \tau_1(\chi_A \otimes 1)$.

The inverse of $[A] \in \mathrm{Gal}(R, H)$ is represented by A^{opp}, with comodule structure map $\chi_{A^{-1}} = (1 \otimes S)\chi_A$.

If $H = (RG)^*$, where G is a finite abelian group, then an H-Galois object is a Galois extension of R in the sense of [17]. In this case, it is well-known that

$$\mathrm{Gal}^s(R, (RG)^*) \cong H^1(R, G).$$

In this section, we will prove a similar result for H-Galois objects. We will see below that a commutative H-Galois object is in fact a twisted form of H. Therefore, in order to give a cohomological description of $\mathrm{Gal}^s(R, (RG)^*)$, it is important to know the comodule algebra isomorphisms of H; in the following theorem, we show that these are determined by the grouplike elements of the dual Hopf algebra H^*.

2.2. *Theorem*

Let H be a finitely generated projective commutative and cocommutative Hopf algebra. Then we have an isomorphism $\alpha : \mathrm{End}_{H\text{-com}}(H), +, \circ \to H^*, +, *$ *(as usual, $*$ denotes the convolution); α and its inverse are given by* $\alpha(\Phi) = \varepsilon\Phi$ *and* $\alpha^{-1}(f) = 1 * \eta f$.

α restricts to isomorphisms of groups:

$$\alpha' : \mathrm{Aut}_{H\text{-com}}(H), \circ \to \mathbb{G}_m(H^*), *$$

$$\alpha'' : \mathrm{Aut}_{R\text{-alg},\, H\text{-com}}(H), \circ \to G(H^*), *$$

Proof. Recall that $\Phi \in \mathrm{End}(H)$ is an H-comodule endomorphism if and only if $(\Phi \otimes 1)\Delta = \Delta\Phi$. First we show that, for all $f \in H^*$, $1 * \eta f$ is an H-comodule endomorphism. Indeed, for all $h \in H$, we have

$$\begin{aligned}\Delta(1 * \eta f)(h) &= \Delta\mu(1 \otimes \eta f)\Delta(h) \\ &= (\mu \otimes \mu)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)(1 \otimes \eta f)\Delta(h) \\ &= (\mu \otimes \mu)(1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes (\eta \otimes \eta)(f))(\Delta \otimes 1)(h) \\ &= \sum_{(h)} h_{(1)}\eta(f(h_{(3)})) \otimes h_{(2)}\end{aligned}$$

and

$$\begin{aligned}((1 * \eta f) \otimes 1)\Delta(h) &= (\mu(1 \otimes \eta f) \otimes 1)(\Delta \otimes 1)\Delta(h) \\ &= \sum_{(h)} h_{(1)}\eta(f(h_{(2)})) \otimes h_{(3)}.\end{aligned}$$

From the cocommutativity of H, it follows that $1 * \eta f$ is an H-comodule endomorphism.

The maps α and α^{-1} defined above are each other's inverse: take $\Phi \in \mathrm{End}_{H\text{-com}}(H)$. Then we have successively

$$(\Phi \otimes 1)\Delta = \Delta\Phi;$$

$$\mu(1 \otimes S)(\Phi \otimes 1)\Delta = \mu(1 \otimes S)\Delta\Phi;$$

$$\Phi * S = (1 * S)\Phi = \eta(\varepsilon\Phi);$$

$$\Phi = 1 * \eta(\varepsilon\Phi).$$

Conversely, take $f \in H^*$; then $\varepsilon(1 * \eta f) = f$. Indeed, for all $h \in H$, we have:

$$\begin{aligned}\varepsilon(1 * \eta f)(h) &= \varepsilon\mu(1 \otimes \eta f)\Delta(h)\\ &= \varepsilon\left(\sum_{(h)} h_{(1)}\eta f(h_{(2)})\right)\\ &= \sum_{(h)} \varepsilon(h_{(1)})\varepsilon\eta f(h_{(2)})\\ &= \sum_{(h)} \varepsilon(h_{(1)}) f(h_{(2)})\\ &= f\left(\sum_{(h)} \varepsilon(h_{(1)})h_{(2)}\right)\\ &= f(h).\end{aligned}$$

It is clear that α preserves the addition; we also have that $\alpha(\Phi \circ \Psi) = \alpha(\Phi) * \alpha(\Psi)$. Indeed, using the facts that $\varepsilon * \varepsilon = \varepsilon$ and $\tau\Delta = \Delta$, we obtain

$$\begin{aligned}\varepsilon\Phi * \varepsilon\Psi &= \mu(\varepsilon \otimes \varepsilon)(\Phi \otimes \Psi)\Delta\\ &= \mu(\varepsilon \otimes \varepsilon)\tau(\Psi \otimes 1)\tau(\Phi \otimes 1)\Delta\\ &= \mu(\varepsilon \otimes \varepsilon)\tau(\Psi \otimes 1)\Delta\Phi\\ &= \mu(\varepsilon \otimes \varepsilon)\tau\Delta\Psi\Phi\\ &= (\varepsilon * \varepsilon)(\Psi\Phi)\\ &= \varepsilon(\Psi\Phi).\end{aligned}$$

If we apply the functor $\mathbb{G}_m$ to α, then we obtain the isomorphism α'.

Next, we show that $\alpha(\Phi) = f$ is grouplike if Φ is an H-comodule algebra isomorphism of H. For all $h, h' \in H$, we have (using that Φ and ε are algebra maps):

$$\mu^*(f)(h \otimes h') = f(hh') = f(h)f(h') = (f \otimes f)(h \otimes h'),$$

so $\mu^*(f) = f \otimes f$. Conversely, if f is grouplike, then $\Phi = 1 * \eta f$ is an algebra automorphism, because η is an algebra homomorphism, and because the convolution of two algebra homomorphisms is an algebra homomorphism. This finishes the proof of the theorem.

2.3. *Lemma*

Suppose that S is a faithfully flat R-algebra, and that A is an H-comodule algebra. If $A \otimes S$ is an $H \otimes S$-Galois object, then A is an H-Galois object.

PROOF. A is faithfully flat as an R-module, because $A\otimes S$ is a faithfully flat S-module, and because A is a faithfully flat R-algebra. Furthermore

$$\gamma_S:(A\otimes S)\otimes_S(\underline{A}\otimes S)=(A\otimes\underline{A})\otimes S$$
$$\to(H\otimes S)\otimes_S(\underline{A}\otimes S)=(H\otimes\underline{A})\otimes S,$$

defined by

$$\gamma_S((x\otimes s)\otimes(y\otimes t))=\sum_{(x)}(x_{(1)}\otimes s)\otimes(x_{(0)}y\otimes t)$$

is an isomorphism. It is easily seen that $\gamma_S=\gamma\otimes 1$, where $\gamma:A\otimes\underline{A}\to H\otimes\underline{A}$. γ is an isomorphism because S is a faithfully flat R-algebra.

2.4. *A comodule version of the Picard group*

Pic(R,H) will denote the set of H-comodule isomorphism classes of H-comodules I such that $I\otimes S\cong H\otimes S$ as $H\otimes S$-comodules, for some faithfully flat R-algebra S. For a fixed faithfully flat R-algebra S, Pic$(S/R,H)$ will be the subset of Pic(R,H) consisting of classes [I] such that $I\otimes S\cong H\otimes S$ as $H\otimes S$-comodules. Observe that the formula for the product of two Galois extensions also defines a product on Pic(R,H) and Pic$(S/R,H)$. Indeed, if I,J determine elements of Pic(R,H), then

$$I\,.\,J=\{x\in I\otimes J:\tau_1(\chi_I\otimes 1)(x)=(1\otimes\chi_J)(x)\}$$

with comodule structure map $\chi_{I.J}$ given by the restriction of the map $\chi_I\otimes 1$ or $1\otimes\chi_J$. Indeed, suppose $I\in$ Pic$(S/R,H)$, $J\in$ Pic$(S'/R,H)$. Replacing S and S' by $S\otimes S'$, we may suppose that $S=S'$. We therefore have that

$$I\,.\,J\otimes S=(I\otimes S)\,.\,(J\otimes S)\cong(H\otimes S)\,.\,(H\otimes S)\cong H\otimes S,$$

so $I\,.\,J\in$ Pic$(S/R,H)$. So we have defined a product on Pic(R,H) and Pic$(S/R,H)$. It is clear that H is a unit element for this product; we will see below that Pic(R,H) and Pic$(S/R,H)$ are (abelian) groups.

Observe that we have a natural map Gal$^s(R,H)\to$ Pic(R,H), induced by the functor forgetting the algebra structure.

In the case where $H=RG$, Pic(R,H) is computed easily. Indeed, suppose that $I\otimes S$ and SG are isomorphic as SG-comodules, i.e. as G-graded S-modules. Then for all $\sigma\in G$, $I_\sigma\otimes S\cong S$, so $I_\sigma\in$ Pic(R). So $I=\oplus_{\sigma\in G}I_\sigma$, with $I_\sigma\in$ Pic(R), and therefore

$$\text{Pic}(R, H) = K^1(G, \text{Pic}(R)),$$

the group of functions from G to $\text{Pic}(R)$.

Let S be a faithfully flat R-algebra. In the terminology of Knus and Ojanguren [21, Sec. 2.8], the elements of $\text{Pic}(S/R, H)$ are the twisted forms of H for the extension S, up to H-comodule isomorphisms. Applying Theorem 2.2, and [21, Prop. II.8.1], we obtain that $\text{Pic}(S/R, H)$ is equal to the Amitsur cohomology group $H^1(S/R, \mathbb{G}_m(H^* \otimes .))$ as a pointed set (one has to make the observation that, in view of Lemma 1.4, one may replace isomorphisms by H-comodule isomorphisms in Knus and Ojanguren's result). A similar observation holds for H-Galois objects:

$$\text{Gal}(S/R, H) = \text{Ker}(\text{Gal}(R, H) \to \text{Gal}(S, H))$$

is equal to $H^1(S/R, G(H^* \otimes .))$ as a pointed set.

Since $\mathbb{G}_m(H^* \otimes S)$ and $G(H^* \otimes S)$ are abelian groups for every faithfully flat extension S of R, $H^1(S/R, \mathbb{G}_m(H^* \otimes .))$ and $H^1(S/R, G(H^* \otimes .))$ are abelian groups. In fact we have:

2.5. *Theorem*

$\text{Pic}(S/R, H)$ *is an abelian group; furthermore, we have the following isomorphisms of groups*:

$$\beta : \text{Gal}(S/R, H) \to H^1(S/R, G(H^* \otimes .))$$

and

$$\gamma : \text{Pic}(S/R, H) \to H^1(S/R, \mathbb{G}_m(H^* \otimes .)).$$

Proof. In view of the remarks preceding the theorem, it suffices to show that β and γ are homomorphisms. Let us show that β is a homomorphism; the proof that γ is a homomorphism is identical to it. Following [21], the map β is defined as follows: suppose that A represents an element of $\text{Gal}(S/R, H)$. Then we have a comodule algebra isomorphism $f: A \otimes S \to H \otimes S$. Define Φ by the commutativity of the following diagram:

$$\begin{array}{ccc} A_{13} & \xrightarrow{f_1} & H_{13} \cong H \otimes S^{(2)} \\ \downarrow \tau_3 & & \downarrow \Phi \\ A_{23} & \xrightarrow[f_3]{} & H_{23} \cong H \otimes S^{(2)} \end{array} \qquad (2.6.1)$$

Φ is a comodule algebra isomorphism of $H \otimes S^{(2)}$, so $\varepsilon\Phi \in G((H \otimes S^{(2)})^*) = G(H^* \otimes S^{(2)})$. $\varepsilon\Phi$ is a cocycle, because Φ is a descent datum.

Now, let $f: A \to C$ and $g: B \to D$ be H-comodule algebra homomorphisms between H-Galois objects. Then $f \otimes g: A \otimes B \to C \otimes D$ restricts to an H-comodule algebra homomorphsism $f . g: A . B \to C . D$. Indeed, take $x \in A . B$. Let us show that $(f \otimes g)(x) \in C . D$, that is

$$\tau_1(\chi_C \otimes 1)(f \otimes g)(x) = (1 \otimes \chi_D)(f \otimes g)(x)$$

or

$$\tau_1(\chi_C f \otimes g)(x) = (f \otimes \chi_D g)(x)$$

or

$$\tau_1((f \otimes 1)\chi_A \otimes g)(x) = (f \otimes (g \otimes 1)\chi_B)(x)$$

or

$$\tau_1(f \otimes 1 \otimes g)(\chi_A \otimes 1)(x) = (f \otimes g \otimes 1)(1 \otimes \chi_B)(x)$$

or

$$(f \otimes g \otimes 1)\tau_1(\chi_A \otimes 1)(x) = (f \otimes g \otimes 1)(1 \otimes \chi_B)(x).$$

This last statement follows from the fact that $x \in A . B$.

Consider the particular case where $A = B = C = D = H$. In this case,

$$H . H = \{\Delta h \in H \otimes H : h \in H\},$$

and $\Delta: H \to H . H$ is a comodule algebra isomorphism. The comodule structure map on $H . H$ is the restriction of $1 \otimes \Delta$. Let us show that in this case $(f . g)\Delta = \Delta(f \circ g)$:

$$\begin{aligned}(f . g)\Delta &= (f \otimes g)\Delta \\ &= (1 \otimes g)(f \otimes 1)\Delta \\ &= \tau(1 \otimes g)\tau\Delta f \\ &= \tau(g \otimes 1)\Delta f \\ &= \tau\Delta g f \\ &= \Delta g f \\ &= \Delta fg.\end{aligned}$$

Let A, B represent elements of Gal$(S/R, H)$. Then we have the commutative diagram (2.6.1), and, similarly, for B:

$$\begin{array}{ccc} B_{13} & \xrightarrow{g_1} & H_{13} \cong H \otimes S^{(2)} \\ \downarrow \tau_3 & & \downarrow \Psi \\ B_{23} & \xrightarrow[g_3]{} & H_{23} \cong H \otimes S^{(2)} \end{array} \tag{2.6.2}$$

Taking the tensor product of (2.6.1) and (2.6.2), and restricting to the product of Galois objects, we obtain:

$$\begin{array}{ccccc} (A \,.\, B)_{13} & \xrightarrow{f_1 \cdot g_1} & (\Delta H)_{13} & & \cong H \otimes S^{(2)} \\ \downarrow \tau_3 & & \downarrow \Phi \,.\, \Psi = \Delta(\Phi\Psi) & & \downarrow \Phi\Psi \\ (A \,.\, B)_{23} & \xrightarrow[f_3 \cdot g_3]{} & (\Delta H)_{23} & & \cong H \otimes S^{(2)} \end{array} \tag{2.6.3}$$

We obtain that $\beta([A \,.\, B]) = \varepsilon(\Phi\Psi) = \varepsilon\Phi * \varepsilon\Psi = \beta([A]) * \beta([B])$.

2.7. *Corollary*

Gal$^s(R, H) \cong H^1(R, G(H^* \otimes .))$ *and* Pic$(R, H) \cong H^1(R, \mathbb{G}_m(H^* \otimes .))$.

Proof. This follows easily after we take the inductive limit over all faithfully flat extensions of R.

2.8. *Corollary*

$H^1(R, G(H^* \otimes .)) \cong \varinjlim H^1(S/R, G(H^* \otimes .))$, *where S runs over all* faithfully projective *R-algebras S.*

Proof. Take $[f] \in H^1(R, G(H^* \otimes .))$, and let S be its corresponding H-Galois object in Gal$^s(R, H)$. Then $\underset{\sim}{S}$ is a faithfully projective R-algebra, and $[f]$ is represented by a cocycle in $H^1(\underset{\sim}{S}/R, G(H^* \otimes .))$, since $S \otimes \underset{\sim}{S} \cong H \otimes \underset{\sim}{S}$.

2.9. *Corollary*

Pic$(R, H) = $ Pic(H^*).

Proof. It is easily seen that

$$\mathrm{Pic}(S/R, H) \cong H^1(S/R, \mathbb{G}_m(H^* \otimes .)) \cong H^1(H^* \otimes S/H^*, \mathbb{G}_m)$$
$$\cong \mathrm{Pic}(H^* \otimes S/H^*).$$

The result then follows after we take inductive limits over all faithfully flat R-algebras S.

An H-Galois object is said to have normal basis if it lies in the kernel of the map $\mathrm{Gal}(R, H) \to \mathrm{Pic}(R, H)$. It has been established by several authors ([4], [23], [27]) that the Galois objects with normal basis are classified by $H^2(H, R, \mathbb{G}_m)$, where $H^2(H, R, \mathbb{G}_m)$ is Sweedler's second cohomology group evaluated at $\mathbb{G}_m$ (some people call this group Harrison's (generalized) cohomology group). Early and Kreimer ([18]) extended this result to a Kummer-type exact sequence. As an application of the foregoing results, we present a cohomological proof of this exact sequence. First, recall Sweedler's complex (cf. [33]):

$$1 \to \mathbb{G}_m(R) \xrightarrow{\Delta^0} \mathbb{G}_m(H^*) \xrightarrow{\Delta^1} \mathbb{G}_m(H^{*(2)}) \xrightarrow{\Delta^2} \cdots$$

where $\Delta^{n-1}(f) = \delta^0(f) * \delta^1(f)^{-1} * \cdots * \delta^n(f)^{\pm 1}$,

$\delta^0(f)(h_1 \otimes \cdots \otimes h_n) = \varepsilon(h_1) f(h_2 \otimes \cdots \otimes h_n)$,

$\delta^i(f)(h_1 \otimes \cdots \otimes h_n) = f(h_1 \otimes \cdots \otimes h_i h_{i+1} \otimes \cdots \otimes h_n)$, for $i = 1, \ldots, n-1$,

and

$\delta^n(f)(h_1 \otimes \cdots \otimes h_n) = \varepsilon(h_n) f(h_1 \otimes \cdots \otimes h_{n-1})$.

Here the inverses are taken with respect to convolution. We see immediately that $\Delta^0(r) = \varepsilon$, for all $r \in R$, since $\delta^0(r)(h) = \delta^0(r)(h) = \varepsilon(h) r$. For $f \in \mathbb{G}_m(H^*)$, we have that $f \in \mathrm{Ker}\, \Delta_1$ if and only if $\delta^0(f) * \delta^2(f) = \delta^1(f)$, or, for all $h, k \in H$:

$$\sum_{(h),(k)} \varepsilon(h_{(1)}) f(k_{(1)}) \varepsilon(k_{(2)}) f(h_{(2)}) = f(hk)$$

or

$$f(h) f(k) = f(hk)$$

or f is grouplike. This fact was already observed by Sweedler in [33]. Therefore, we have that $H^1(H, R) = Z^1(H, R) = G(H^*)$, and we have a short exact sequence

$$1 \to G(H^*) = H^1(H, R) \to \mathbb{G}_m(H^*) \to Z^2(H, R).$$

Let us call $f \in H^{*(2)}$ symmetric if $f = f \circ \tau$. The subgroup of $Z^2(H, R)$ consisting of symmetric cocycles is denoted by $Z^2_{\mathrm{symm}}(H, R)$.

It is clear that $\operatorname{Im} \Delta^1$ is contained in $Z^2_{\mathrm{symm}}(H, R)$.

2.10. *Proposition*

We have an exact sequence of sheaves on R_{fl}

$$1 \to G(H^* \otimes .) \to \mathbb{G}_m(H^* \otimes .) \to Z^2_{\mathrm{symm}}(H \otimes .,.) \to 1.$$

Proof. We have to show that for all $f \in Z^2_{\mathrm{symm}}(H \otimes .,.)$, there exists a faithfully flat R-algebra S, such that $[f_S] = [f \otimes 1_S] = 1$ in $H^2_{\mathrm{symm}}(H \otimes S, S)$. For S, we take the following R-algebra: as an R-module we take S equal to H. The element of S corresponding to $h \in H$ will be denoted by u_h. The multiplication on S will be defined by

$$u_h u_k = \sum_{(h),(k)} f(h_{(1)} \otimes k_{(1)}) u_{h_{(2)}k_{(2)}};$$

S is associative because f is a cocycle (in fact, $S = R \#_f H$ in the notation of Sweedler), and S is commutative, because f is symmetric. The cocycle

$$f_S = f \otimes 1_S : (H \otimes S) \otimes_S (H \otimes S) = (H \otimes H) \otimes S \to S$$

is given by the formula

$$f_S((h \otimes k) \otimes u_m) = f(h \otimes k) u_m.$$

Define $g : H \otimes S \to S$ by $g(h \otimes u_m) = u_h u_m$. We claim that $f_S = \Delta^1 g$. To this end, we have to show that $f_S * \delta^1(g) = \delta^0(g) * \delta^2(g)$, or, for all $h, k, m, n \in H$:

$$\sum_{(h),(k)} f((h_{(1)} \otimes k_{(1)}) \otimes 1) g(h_{(2)}k_{(2)} \otimes u_m u_n) = g(h \otimes u_m) g(h \otimes u_n)$$

or

$$\sum_{(h),(k)} f((h_{(1)} \otimes k_{(1)}) \otimes 1) u_{h_{(2)}} u_{k_{(2)}} u_m u_n = u_h u_m u_k u_n.$$

This follows from the multiplication rule for S.

2.11. *Corollary* (*Kummer's exact sequence for H-Galois objects*)

We have an exact sequence

$$1 \to H^2_{\mathrm{symm}}(H, R) \to \mathrm{Gal}^s(R, H) \to Z^1(H, R, \mathrm{Pic}).$$

Consequently, every Galois object has normal basis if $\mathrm{Pic}(H^*) = 1$.

Proof. The short exact sequence of sheaves in Proposition 2.10 results in a long exact sequence of abelian groups

$$1 \to G(H^*) \to \mathbb{G}_m(H^*) \to Z^2_{\mathrm{symm}}(H, R, \mathbb{G}_m)$$

$$\to H^1(R, G(H^* \otimes .)) = \mathrm{Gal}^s(R, H)$$

$$\to H^1(R, \mathbb{G}_m(H^* \otimes .)) = \mathrm{Pic}(H^*)$$

$$\to H^1(R, Z^2_{\mathrm{symm}}(H \otimes ., ., \mathbb{G}_m)) \subset K^2_{\mathrm{symm}}(H, R, \mathrm{Pic}) \to \cdots$$

2.11 follows immediately from the exactness of this sequence.

2.12. *Note*

The image of $[f] \in H^2_{\mathrm{symm}}(H, R)$ is the R-algebra S constructed in the proof of Proposition 2.10, with comodule structure map

$$\chi_S = (\alpha \otimes 1)\Delta\alpha^{-1}$$

where α is the R-module isomorphism $\alpha : H \to S$ given by $\alpha(h) = u_h$. In other words:

$$\chi_S(u_h) = \sum_{(h)} u_{h_{(1)}} \otimes h_{(2)}.$$

3. The split part of the Brauer–Long group

Let H be a commutative and cocommutative faithfully projective Hopf algebra over the commutative ring R. For an H-dimodule algebra A, we define the maps

$$F : A \# \bar{A} \to \mathrm{End}_R(A),$$

$$G : \bar{A} \# A \to \mathrm{End}_R(A)^{\mathrm{opp}}$$

by

$$F(a \# \bar{b})(c) = \sum_{(b)} a(b_{(1)} \rightharpoonup c)b_{(0)},$$

$$G(\bar{a} \# b)(c) = \sum_{(c)} (c_{(1)} \rightharpoonup a)c_{(0)}b.$$

It may be verified (cf. [25, prop. 4.1]) that F and G are H-dimodule algebra homomorphisms. Recall the following definition from [25]:

3.1. *Definition*

An H-dimodule algebra A is called an H-Azumaya algebra if and only if A is faithfully projective as an R-module and the maps F and G defined above are isomorphisms of H-dimodule algebras.

For example, if $P \in \mathbf{FPD}(R, H)$, then $\mathrm{End}_R(P)$, with the induced dimodule algebra structure, is an H-Azumaya algebra.

3.2. *Definition* (Long [25])

Two H-Azumaya algebras A and B are called Brauer equivalent as H-dimodule algebras (we denote this by $A \sim B$) if there exists $P, Q \in \mathbf{FPD}(R, H)$, such that

$$A \# \mathrm{End}_R(P) \cong B \# \mathrm{End}_R(Q)$$

as H-dimodule algebras.

3.3. *Proposition* (Long [25])

$\sim$ *defines an equivalence relation on the set of H-dimodule algebra isomorphism classes of H-Azumaya algebras. The quotient set is a group under the multiplication induced by* $\#$. *The inverse class of* $[A]$ *is represented by* $[\bar{A}]$. *This group is called the Brauer–Long group of H-dimodule algebras (following the terminology introduced by DeMeyer and Ford* [16]), *and is denoted by* $\mathrm{BD}(R, H)$.

Long showed that $\mathrm{BD}(\,.\,, H \otimes .\,)$ is a covariant functor from commutative R-algebras to groups. For S a commutative faithfully flat R-algebra, we write

$$\mathrm{BD}(S/R, H) = \mathrm{Ker}(\mathrm{BD}(R, H) \to \mathrm{BD}(S, H \otimes S));$$

$$\mathrm{BD}^s(R, H) = \bigcup\nolimits_S \mathrm{BD}(S/R, H),$$

where the union is taken over all faithfully flat R-algebras S. In general, $\mathrm{BD}^s(R, H)$ does not cover the full Brauer–Long group; in the case where $H = RG$, for G a finite abelian group, this is well-known. In Section 4, we will give an example of a Hopf algebra, which is not a group ring, for which $\mathrm{BD}^s(R, H) \neq \mathrm{BD}(R, H)$. The main objective of this paper is to give a cohomological description of $\mathrm{BD}^s(R, H)$. In view of some previous results (cf. [5, 8, 10, 16, . . .]), it will be no surprise that our description will be highly related to Galois theory. It is known that the smash product of an H-Galois object and an H^*-Galois object is an Azumaya algebra. This yields a homomorphism

$$\psi : \mathrm{Gal}^s(R,H) \times \mathrm{Gal}^s(R,H^*) \to \mathrm{Br}(R).$$

In the sequel, we will need a description of ψ on the cocycle level. Let S be a faithfully flat R-algebra, and consider the map

$$\varphi : G(H^* \otimes S^{(2)}) \times G(H \otimes S^{(2)}) \to \mathbb{G}_m(S^{(3)})$$

defined by

$$\varphi(h^*, h) = h_1^*(h_3).$$

3.4. *Theorem*

The map φ induces a map

$$\varphi : H^1(S/R, G(H^* \otimes .)) \times H^1(S/R, G(H \otimes .)) \to H^2(S/R, \mathbb{G}_m). \tag{3.4.1}$$

After taking inductive limits over all commutative faithfully flat R-algebras S, we obtain a map

$$\varphi : H^1(R, G(H^* \otimes .)) \times H^1(R, G(H \otimes .)) \to H^2(R, \mathbb{G}_m). \tag{3.4.2}$$

Furthermore, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Gal}(S/R, H) \times \mathrm{Gal}(S/R, H^*) & \xrightarrow{\Psi} & \mathrm{Br}(S/R) \to \mathrm{Br}(R) \\ \downarrow (\alpha, \alpha^*) & & \downarrow \beta \\ H^1(R, G(H^* \otimes .)) \times H^1(R, G(H \otimes .)) & \xrightarrow{\varphi} & H^2(R, \mathbb{G}_m) \end{array} \tag{3.4.3}$$

Here α and α^* are the isomorphisms of Corollary 2.7. *β is the well-known embedding* $\mathrm{Br}(R) \to H^2(R, \mathbb{G}_m)$ (cf. [21, 34], for example), *and ψ is defined by*

$$\psi([T], [U]) = [T \# U],$$

where the smash product is taken with respect to the H^*-module structure of T and the H^*-comodule structure of U.

PROOF. Let S be a faithfully flat R-algebra, and take $h \in Z^1(R, G(H \otimes .))$, $h^* \in Z^1(R, G(H^* \otimes .))$. Therefore, $h \in G(H \otimes S \otimes S)$ and $h^* \in G(H^* \otimes S \otimes S)$ satisfy $h_1 h_3 = h_2$, and $h_1^* h_3^* = h_2^*$. We will show that $\varphi(h^*, h) \in \mathbb{G}_m(S \otimes S \otimes S)$ is a cocycle in $Z^2(S/R, \mathbb{G}_m)$.

Indeed

$$\begin{aligned}\Delta_2\varphi(h^*, h) &= (h_1^*(h_3))_1(h_1^*(h_3))_2^{-1}(h_1^*(h_3))_3(h_1^*(h_3))_4^{-1}\\ &= h_{11}^*(h_{31})h_{12}^*(h_{32}^{-1})h_{13}^*(h_{33})h_{14}^{*-1}(h_{44})\\ &= h_{11}^*(h_{31}h_{32}^{-1})(h_{13}^*h_{14}^{*-1})(h_{34})\\ &= h_{11}^*((h_1h_2^{-1})_4)((h_2^*h_3^{*-1})_1)(h_{34})\\ &= h_{11}^*(h_{34}^{-1})h_{11}^*(h_{34})\\ &= 1,\end{aligned}$$

Furthermore, if h or h^* is a coboundary, then $\varphi(h^*, h)$ is also a coboundary. Indeed, suppose $h \in B^1(S/R, G(H \otimes .))$, or $h = k_1k_2^{-1}$ for some $k \in G(H \otimes S)$. Then

$$\begin{aligned}\Delta_1(h^*(k_2)) &= (h^*(k_2))_1(h^*(k_2))_2^{-1}(h^*(k_2))_3\\ &= h_1^*(k_{21})h_2^{*-1}(k_{22})h_3^*(k_{23})\\ &= h_1^*(k_{13})(h_2^{*-1}h_3^*)(k_{23})\\ &= h_1^*(k_{13}k_{23}^{-1})\\ &= h_1^*(h_3)\\ &= \varphi(h^*, h).\end{aligned}$$

We therefore have a well-defined map between Amitsur cohomology groups, and, taking inductive limits, we obtain a well-defined map between flat cohomology groups. Before showing the commutativity of the diagram (3.4.1), we remark that $h_1^*(h_3)$ and $h_3^*(h_1)^{-1}$ are cohomologous cocycles. Indeed,

$$\begin{aligned}\Delta_1(h^*(k_2)) &= h_1^*(h_1)h_2^*(h_2)^{-1}h_3^*(h_3)\\ &= h_1^*(h_1)(h_1^*h_3^*)(h_1^{-1}h_3^{-1})h_3^*(h_3)\\ &= h_1^*(h_3)h_3^*(h_1).\end{aligned}$$

Furthermore, observe that H as an H-comodule algebra is an H-Galois object. Therefore we have an isomorphism j of R-algebras ([12, Th. 9.3]):

$$j: H \# H^* \to \mathrm{End}_R(H)$$

given by

$$j(h \# h^*)(x) = \sum_{(x)} h^*(x_{(1)})hx_{(2)}.$$

Let $\Phi: H \to H$ and $\Psi: H^* \to H^*$ be respectively an H-comodule algebra isomorphism and an H^*-comodule algebra isomorphism. By Theorem 2.3, there exists $u \in G(H^*)$ and $g \in G(H)$ such that

$$\Phi = 1 * u, \quad \text{or} \quad \Phi(x) = \sum_{(x)} x_{(1)} u(x_{(2)}),$$

$$\Psi = 1 * g \quad \text{or} \quad \Psi(x^*) = \sum_{(x^*)} x^*_{(1)}(g) x^*_{(2)}$$

$$\text{or} \quad \Psi(x^*)(x) = \sum_{(x^*)} x^*_{(1)}(g) x^*_{(2)}(x) = x^*(gx)$$

for all $x \in H$, $x^* \in H^*$.

Define $\Theta : \mathrm{End}_R(H) \to \mathrm{End}_R(H)$ by commutativity of the following diagram:

$$\begin{array}{ccc} H \# H^* & \xrightarrow{j} & \mathrm{End}_R(H) \\ \downarrow \Phi \# \Psi & & \downarrow \Theta \\ H \# H^* & \xrightarrow[j]{} & \mathrm{End}_R(H) \end{array}$$

Define $f : H \to H$ by $f(x) = \Sigma_{(x)} u(Sx_{(1)}) g x_{(2)}$. Then Θ is induced by f. To prove this, we have to show that for all $\lambda \in \mathrm{End}_R(H)$:

$$\Theta(\lambda) = f^{-1} \circ \lambda \circ f.$$

It suffices to show this for $\lambda = j(h \# h^*)$, for all $h \in H$ and $h^* \in H^*$. Observe that

$$f^{-1}(x) = \sum_{(x)} u(g^{-1} x_{(1)}) g^{-1} x_{(2)}.$$

For all $x \in H$, we have

$$\lambda(f(x)) = \sum_{(x)} u(Sx_{(1)}) h^*(gx_{(2)}) g h x_{(3)}$$

and

$$\begin{aligned} f^{-1}(\lambda(f(x))) &= \sum_{(x),(h)} u(Sx_{(1)}) h^*(gx_{(2)}) u(g^{-1} g h_{(1)} x_{(3)}) g^{-1} g h_{(2)} x_{(4)} \\ &= \sum_{(x),(h)} u(h_{(1)}) h^*(gx_{(1)}) h_{(2)} x_{(2)}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\Theta(\lambda)(x) &= j(\Phi(h)\#\Psi(h^*))(x)\\ &= \sum_{(x)} \Psi(h^*)(x_{(1)})\Phi(h)x_{(2)}\\ &= \sum_{(x),(h)} h^*(gx_{(1)})u(h_{(1)})h_{(2)}x_{(2)}.\end{aligned}$$

proving the assertion.

With these necessary preparations done, we are able to prove the commutativity of the diagram (3.4.3). Take $[T]\in \mathrm{Gal}(S/R, H)$, $[U]\in \mathrm{Gal}(S/R, H^*)$. Using Theorem 2.6, we obtain the following commutative diagrams:

$$\begin{array}{ccccccc} T_{13} & \xrightarrow{\sigma_1} & H_{13} & \quad & U_{13} & \xrightarrow{\sigma_1^*} & H_{13}^* \\ \downarrow \tau_3 & & \downarrow \Phi & & \downarrow \tau_3 & & \downarrow \Psi \\ T_{23} & \xrightarrow[\sigma_3]{} & H_{23} & & U_{23} & \xrightarrow[\sigma_3^*]{} & H_{23}^* \end{array}$$

Taking the smash product of these two diagrams, we obtain:

$$\begin{array}{ccc} (T\#U)_{13} \xrightarrow{(\sigma\otimes\sigma^*)_1} & (H\#H^*)_{13} = (H\#H^*)\otimes S^{(2)} \rightarrow & \mathrm{End}_{S^{(2)}}(H\otimes S^{(2)}) \\ \downarrow \tau_3 & \downarrow \Phi\#\Psi & \downarrow \Theta \\ (T\#U)_{23} \xrightarrow{(\sigma\otimes\sigma^*)_3} & (H\#H^*)_{23} = (H\#H^*)\otimes S^{(2)} \rightarrow & \mathrm{End}_{S^{(2)}}(H\otimes S^{(2)}). \end{array}$$

Let $u\in G(H^*\otimes S^{(2)})$, $g\in G(H\otimes S^{(2)})$ be the grouplike elements corresponding to Φ and Ψ. Then $\alpha([T]) = [u]$ and $\alpha^*([U]) = g$. Since Θ is induced by

$$f: H\otimes S^{(2)} \rightarrow H\otimes S^{(2)}$$

given by

$$f(x) = \sum_{(x)} u(Sx_{(1)})gx_{(2)}$$

we have that, for all $y\in H\otimes S^{(2)}$:

$$\begin{aligned} f_2^{-1}\circ f_3\circ f_1(y) &= f_2^{-1}\circ f_3\left(\sum_{(y)} u_1(S(y_{(1)})g_1 y_{(2)}\right)\\ &= f_2^{-1}\left(\sum_{(y)} u_1(S(y_{(1)})u_3(S(g_1 y_{(2)}))g_3 g_1 y_{(3)}\right)\\ &= \sum_{(y)} u_1(S(y_{(1)})u_3(S(g_1 y_{(2)}))u_2(g_2^{-1}g_3 g_1 y_{(3)})g_2^{-1}g_3 g_1 y_{(4)} \end{aligned}$$

$$= \sum_{(y)} u_1(S(y_{(1)})u_3(g_1^{-1}S(y_{(2)})u_1(y_{(3)})u_3(y_{(4)})y_{(5)}$$

$$= \sum_{(y)} u_1(y_{(3)}S(y_{(1)})u_3(g_1^{-1}y_{(4)}S(y_{(2)})y_{(5)}$$

$$= u_3(g_1^{-1})y.$$

Therefore the image of $[T\# U]$ in $H^2(R, \mathbb{G}_m)$ is given by $[u_3(g_1^{-1})] = \varphi(u, g)$. This finishes the proof of Theorem 3.4.

3.5. *Note*

In (3.4.3), the reader might prefer to have a smash product with respect to H-module and comodule structure rather than H^*-module and comodule structure. This may be obtained easily: we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(S/R, H^*) \times \mathrm{Gal}(S/R, H) \xrightarrow{\psi'} \mathrm{Br}(S/R) \longrightarrow & \mathrm{Br}(R) \\ \downarrow (\alpha^*, \alpha) & \downarrow \beta \\ H^1(R, G(H\otimes .)) \times H^1(R, G(H^*\otimes .)) & \xrightarrow{\varphi'} H^2(R, \mathbb{G}_m) \end{array}$$

where ψ' is defined as ψ, but with respect to the H-module and comodule structure, and where φ' is defined by $\varphi'(g, u) = \varphi(u, g)^{-1}$. Indeed, it may be shown easily that, for $T \in \mathrm{Gal}(R, H)$, $U \in \mathrm{Gal}(R, H^*)$, $T\# U \cong (U\# T)^{\mathrm{opp}}$.

3.6. *A generalization of Villamayor and Zelinsky's E_1 and E_2*

Villamayor and Zelinsky defined abelian groups E_i such that, for S a faithfully flat R-algebra, we have a long exact sequence

$$0 \to H^1(S/R, \mathbb{G}_m) \to E_1 \to H^0(S/R, \mathrm{Pic}) \to \cdots$$
$$\to H^i(S/R, \mathbb{G}_m) \to E_i \to H^{i-1}(S/R, \mathrm{Pic}) \to \cdots .$$

For details, we refer to [38, 40]. It may be shown that $\mathrm{Br}(S/R)$ is a subgroup of E_2, and that $H^1(R, \mathbb{G}_m) \cong \bigcup_S E_i$, where the union is taken over all faithfully flat extensions S of R. In the sequel, the E_i's of Villamayor and Zelinsky will be denoted by E_i^{class}. Our aim is to introduce a generalization of E_1 and E_2, related to the Brauer–Long group. Define a functor

$$\delta_{n-1} : \mathbf{PD}(S^{(n)}, H\otimes S^{(n)}) \to \mathbf{PD}(S^{(n+1)}, H\otimes S^{(n+1)})$$

as follows:

$$\delta_{n-1}(I) = I_1 \otimes_{n+1} I_2^* \otimes_{n+1} \cdots \otimes_{n+1} I_{n+1}^{(*)}.$$

Since $\Delta_n \circ \Delta_{n-1} : S^{(n)} \to S^{(n+2)}$ is the zero map, it follows easily that we have a natural H-dimodule isomorphism

$$\lambda_I : \delta_n \delta_{n-1}(I) \to S^{(n+2)}.$$

The isomorphism λ_I is determined by the canonical isomorphism $I \otimes I^* \cong S^{(n)}$. For example,

$$\delta_1\delta_0(I) = (I_{11} \otimes_3 I_{21}^*) \otimes_3 (I_{12} \otimes_3 I_{22}^*)^* \otimes_3 (I_{13} \otimes_3 I_{23}^*)$$

$$= (I_{11} \otimes_3 I_{12}^*) \otimes_3 (I_{13} \otimes_3 I_{21}^*) \otimes_3 (I_{22} \otimes_3 I_{23}^*) \cong S^{(3)}.$$

Identifying $I \otimes I^*$ and $S^{(n)}$, we will often view λ_I as the identity.

Let Ω_n be the category with objects (I, α), where

$$I \in \mathbf{PD}(S^{(n)}, H \otimes S^{(n)}),$$

and

$$\alpha : \delta_{n-1}(I) \to S^{(n+1)}$$

is an $H \otimes S^{(n+1)}$-dimodule isomorphism such that $\delta_n \alpha = \lambda_I$. A morphism between two objects (I, α) and (J, β) of Ω_n is $H \otimes S^{(n)}$-dimodule isomorphism $f: I \to J$ such that

$$\alpha = \beta \circ \delta_{n-1} f.$$

Let Z_n be the set of isomorphism classes of Ω_n. $E_1 = Z_1$ is a group, with multiplication induced by the tensor product. On Z_2, we define a twisted multiplication as follows:

$$[(I, \alpha)][(J, \beta)] = [(I \otimes_2 J, \varphi(e^*([I]), e([J]))(\alpha \otimes \beta))],$$

where

$$e^* : \mathbf{PD}(S^{(2)}, H \otimes S^{(2)}) \to G(H^* \otimes S^{(2)}),$$

$$e : \mathbf{PD}(S^{(2)}, H \otimes S^{(2)}) \to G(H \otimes S^{(2)})$$

are the projections discussed in (1.6.3). It then follows easily that Z_2 is a group. B_2 will be the subgroup of Z_2 consisting of all elements of the form $[(\delta_0 J, \lambda_J)]$; we define $E_2 = Z_2/B_2$.

3.7. *Lemma*

(3.7.1) $E_1 \cong G(H^*) \times G(H) \times E_1^{\text{class}}$;
(3.7.2) $E_2 \cong H^1(S/R, G(H^* \otimes .)) \times H^1(S/R, G(H \otimes .)) \times_\varphi E_2^{\text{class}}$,
where the multiplication on the twisted product in (3.7.2) *is given by*

$$([g^*], [g], [I, \alpha])([h^*], [h], [J, \beta]) = ([g^*h^*], [gh], [I \otimes J, \varphi(g^*, h)\alpha \otimes \beta]).$$

PROOF. (3.7.1) is obvious. To show (3.7.2), we define a map

$$\mu : E_2 \to H^1(S/R, G(H^* \otimes .)) \times H^1(S/R, G(H \otimes .)) \times_\varphi E_2^{\text{class}},$$

by $\mu([I, a]) = ([e^*(I)], [e(I)], [(\underline{I}, a)])$. It is straightforward to verify that μ is an isomorphism; the inverse of μ is given by

$$\mu^{-1}([g^*], [g], [I, \alpha]) = [I(g^*, g), \alpha].$$

3.8. *Theorem*

Let S be a faithfully flat R-algebra. Then $E_1 \cong \mathrm{PD}(R, H)$, *and we have a monomorphism*

$$i : \mathrm{BD}(S/R, H) \to E_2$$

which is an isomorphism if S is faithfully projective as an R-module.

PROOF. The statement for E_1 follows from (3.7.1) and (1.6.3). The proof of the second part of the theorem consists of several steps; only one of them is significantly different from Villamayor and Zelinsky's corresponding theorem for the Brauer group.

(3.8.1) Take $[A] \in \mathrm{BD}(S/R, H)$. We then have a dimodule algebra isomorphism

$$\rho : A \otimes S \to \mathrm{End}_S(Q).$$

for some $Q \in \mathrm{FPD}(S, H \otimes S)$. Define a dimodule algebra Φ by commutativity of the following diagram:

$$\begin{array}{ccc} A_{13} & \xrightarrow{\rho_1} & \mathrm{End}_2(Q_1) \\ \downarrow \tau_3 & & \downarrow \Phi \\ A_{23} & \xrightarrow[\rho_3]{} & \mathrm{End}_2(Q_2) \end{array}$$

Applying Proposition 1.8, we see that Φ is induced by a dimodule isomorphism

$$f: Q_1 \otimes_2 I \to Q_2,$$

for some $I \in \mathbf{PD}(S^{(2)}, H \otimes S^{(2)})$. Now

$$(f_2^{-1} \otimes I_2) \circ (f_3 \otimes I_2^*) \circ (f_1 \otimes I_3 \otimes I_2^*) : Q_{11} \otimes_3 I_1 \otimes_3 I_3 \otimes_3 I_2^* \to Q_{11}$$

induces $\Phi_2^{-1}\Phi_3\Phi_1 = 1$. To simplify notations, we will also use the — however incorrect — notation

$$(f_2^{-1} \otimes I_2^*) \circ (f_3 \otimes I_2^* \circ (f_1 \otimes I_3 \otimes I_2^*) = f_2^{-1} f_3 f_1.$$

From the uniqueness property in Proposition 1.8, it follows that we have an $H \otimes S^{(3)}$-dimodule isomorphism

$$\alpha : \delta_1(I) = I_1 \otimes_3 I_3 \otimes_3 I_2^* \to S^{(3)}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} Q_{11} \otimes_3 \delta_1(I) & \xrightarrow{f_2^{-1} f_3 f_1} & Q_{11} \\ \downarrow (1 \otimes \alpha) & & \downarrow = \\ Q_{11} \otimes_3 S^{(3)} & \xrightarrow[\mathrm{Id}]{} & Q_{11} \end{array}$$

Therefore $f_2^{-1} f_3 f_1 = 1 \otimes \alpha$.

(3.8.2) (I, α) represents an element of Ω_2.

Indeed, $f_2^{-1} f_3 f_1 = 1 \otimes \alpha$, so $1 \otimes \delta_2 \alpha = \delta_2(f_2^{-1} f_3 f_1) = 1 \otimes \lambda_I$, so $\alpha = \lambda_I$. Write $\theta(A, \rho) = [I, \alpha] \in E_2$. Applying the uniqueness property in 1.8, we may easily see that $[I, \alpha]$ is independent of the choice of f and I.

(3.8.3) $\theta(A, \rho) = [I, \alpha]$ is independent of the choice of ρ.

Let $\rho' : A \otimes S \to \mathrm{End}_S(Q')$ be another choice. Then $\rho'\rho^{-1} : \mathrm{End}_S(Q) \to \mathrm{End}_S(Q')$ is induced by an H-dimodule isomorphism $g : Q \otimes J \to Q'$, for some $J \in \mathbf{PD}(S, H \otimes S)$. Consider the commutative diagram

$$\begin{array}{ccccc} A_{13} & \xrightarrow{\rho_1} & \mathrm{End}_2(Q_1) & \xrightarrow{(\rho'\rho^{-1})_1} & \mathrm{End}_2(Q_1') \\ \downarrow \tau_3 & & \downarrow \Phi & & \downarrow \Phi' \\ A_{23} & \xrightarrow[\rho_3]{} & \mathrm{End}_2(Q_2) & \xrightarrow[(\rho'\rho^{-1})_2]{} & \mathrm{End}_2(Q_2') \end{array}$$

Then Φ' is induced by

$$f' = g_2 \circ (f \otimes J_2) \circ (g_1^{-1} \otimes J_1^* \otimes I \otimes J_2) : Q_1' \otimes_2 J_1^* \otimes_2 I \otimes_2 J_2 \to Q_2',$$

or, by the same abuse of notation mentioned above:

$$f' = g_2 \circ f \circ g_1^{-1}.$$

Observe that

$$(g^{-1} \otimes J^*) \circ g : Q \otimes J \otimes J^* \to Q$$

is given by

$$(g^{-1} \otimes J^*) \circ g = 1 \otimes \mu,$$

where μ is the canonical morphism $\mu : J \otimes J^* \to S$. Now

$$\begin{aligned} f_2'^{-1} f_3' f_1' &= (g_{12} f_2^{-1} g_{22}^{-1})(g_{23} f_3 g_{13}^{-1})(g_{21} f_1 g_{11}^{-1}) \\ &= g_{12}(f_2^{-1} f_3 f_1)(g_{22}^{-1} g_{23})(g_{13}^{-1} g_{21}) g_{11}^{-1} \\ &= g_{12}(1 \otimes \alpha)(1 \otimes \mu_{22})(1 \otimes \mu_{13}) g_{11}^{-1} \\ &= (1 \otimes \alpha) g_{12} g_{11}^{-1} (1 \otimes \mu_{22})(1 \otimes \mu_{13}) \\ &= (1 \otimes \alpha)(1 \otimes \lambda_{J^*}). \end{aligned}$$

We used the fact that

$$\lambda_{J^*} : \delta_1 \delta_0(J^*) = (J_{11}^* \otimes_3 J_{12}) \otimes_3 (J_{21}^* \otimes_3 J_{13}) \otimes_3 (J_{22}^* \otimes_3 J_{23}) \to S^{(3)}$$

is given by

$$\lambda_{J^*} = \mu_{11} \otimes \mu_{21} \otimes \mu_{22}.$$

It now follows that $(I', \alpha') = (I, \alpha)(\delta_0(J^*), \lambda_{J^*})$.

(3.8.4) If $A = \mathrm{End}_R(P)$ for some $P \in \mathbf{FPD}(R, H)$, then $\theta(A, \rho) = [(S^{(2)}, \lambda_S)]$.

This is obvious: choose for ρ the natural map $\rho : \mathrm{End}_R(P) \otimes S \to \mathrm{End}_S(P \otimes S)$. It follows immediately that Φ is then induced by the identity.

(3.8.5) $\theta(A \# A', \rho \# \rho') = \theta(A, \rho)\theta(A', \rho')$.

This is the most interesting part of the proof, because here the twisted product defined on E_2 comes into the picture. Suppose $\theta(A, \rho) = [(I, \alpha)]$, $\theta(A', \rho') = [(I', \alpha')]$, $\theta(A \# A', \rho \# \rho') = (J, \beta)$. Let Φ, f, Q be defined as above, and introduce similar morphisms and modules Φ', f, Q' for A'. Then $\Phi \# \Phi'$ is induced by

$$f\#f' : (Q_1 \otimes_2 I) \otimes_2 (Q_1' \otimes_2 I') \to Q_2 \otimes_2 Q_2',$$

so it follows immediately that $J = I \otimes_2 I'$. Our main problem is to compute

$$(f\#f')_2^{-1} \circ (f\#f')_3 \circ (f\#f')_1.$$

Recall that

$$\mathrm{PD}(S^{(2)}, H \otimes S^{(2)}) \cong \mathrm{Pic}(S^{(2)}) \times G(H^* \otimes S^{(2)}) \times G(H \otimes S^{(2)}).$$

Let $(\underline{I}, g^*, g)$ be the element corresponding to I. Consider the identity

$$\gamma : \underline{I} \to I.$$

We may easily verify that (observe that $\mathrm{Hom}(\underline{I}, I)$ is itself an H-dimodule)

$$\chi(\gamma) = \gamma \otimes \chi \quad \text{and} \quad h \rightharpoonup \gamma = g^*(h)\gamma$$

for all $h \in H$. Now the map

$$\underline{f} = f \circ (1 \otimes \gamma) : Q_1 \otimes_2 \underline{I} \to Q_2$$

clearly also induces Φ. The important difference between f and $\underline{f}$ is that $\underline{f}$ is not an H-dimodule structure preserving morphism. $\Phi\#\Phi'$ is induced by $\underline{f}\#\underline{f}'$, and

$$(\underline{f}\#\underline{f}')_2^{-1} \circ (\underline{f}\#\underline{f}')_3 \circ (\underline{f}\#\underline{f}')_1$$

may be written as a morphism

$$(\underline{f}\#\underline{f}')_2^{-1} \circ (\underline{f}\#\underline{f}')_3 \circ (\underline{f}\#\underline{f}')_1 : (Q \otimes Q')_{11} \otimes_2 \delta_1(\underline{I}) \to (Q \otimes Q')_{11}.$$

Indeed, since the dimodule structure on $\underline{I}$ is the trivial one, there is no problem in interchanging the I's and Q's. We have

$$\begin{aligned}
(\underline{f}\#\underline{f}')_2^{-1} \circ (\underline{f}\#\underline{f}')_3 \circ (\underline{f}\#\underline{f}')_1 &= g_2^*(g_2')(\underline{f}_2^{-1} \# \underline{f}_2'^{-1}) g_1^*(g_3')(\underline{f}_3\, \underline{f}_1 \# \underline{f}_3'\, \underline{f}_1') \\
&= g_2^*(g_2') g_1(g_3')(g_3 g_1)(g_2'^{-1})(\underline{f}_2^{-1} \underline{f}_3\, \underline{f}_1 \# \underline{f}_2^{-1}\, \underline{f}_3\, \underline{f}_1) \\
&= g_1^*(g_3')(1 \otimes \alpha)(1 \otimes \alpha') \\
&= \varphi(g^*, g')(1 \otimes \alpha \otimes \alpha')
\end{aligned}$$

so $\beta = (\alpha \otimes \alpha')\varphi(g^*, g)$, and this proves the assertion.

From (3.8.4) and (3.8.5), it follows that $\theta(A, r)$ is independent of the choice of A in $[A]$, so we have defined a well-defined homomorphism

$$i : \mathrm{BD}(S/R, H) \to E_2$$

by

$$i([A]) = \theta(A, r).$$

(3.8.6) i is injective.

Suppose $\theta(A,\rho) = (\delta_0 J, \lambda_J)$. Then Φ is induced by $f: Q_1 \otimes_2 J_1 \otimes_2 J_2^* \rightarrow Q_2$. Consider

$$f \otimes J_2 : Q_1 \otimes_2 J_1 \rightarrow Q_2 \otimes_2 J_2.$$

$f \otimes J_2$ is clearly a descent datum, so there exists $P \in \mathbf{FPD}(R, H)$ such that the following diagram is commutative:

$$\begin{array}{ccc} P_{13} & \longrightarrow & (Q \otimes J)_1 \\ \downarrow \tau_3 & & \downarrow f \otimes J_2 \\ P_{23} & \longrightarrow & (Q \otimes J)_2 \end{array}$$

Taking End of this diagram, and invoking the uniqueness property in the theorem of faithfully flat descent, we obtain that $A \cong \mathrm{End}_R(P)$ as H-dimodule algebras.

(3.8.7) If S/R is faithfully projective, then i is surjective.

Take $(I, \alpha) \in E_2$, and consider the dimodule isomorphism

$$\beta = \alpha \otimes I_2 : I_1 \otimes I_3 \rightarrow I_2.$$

Let P be equal to I as an R-module, and furnish P with an $H \otimes S$-dimodule structure as follows:

S-module structure: $s \,.\, x = (s \otimes 1)x$,

$H \otimes S$-module structure: $(h \otimes s) \rightarrow x = (h \otimes s \otimes 1) \rightarrow x$,

$H^* \otimes S$-module structure, and therefore $H \otimes S$-comodule structure:

$$(h^* \otimes s) \rightarrow x = (h^* \otimes s \otimes 1) \rightarrow x.$$

With this structure, P becomes an object of $\mathbf{FPD}(S, H \otimes S)$. We will say that "$P = I$, with S acting on the first factor". We then have:

$P_1 = I_1$, with $S^{(2)}$ acting on the first two factors;

$P_2 = I_3$, with $S^{(2)}$ acting on the first and third factor;

$P_1 \otimes_2 I = I_1 \otimes_3 I_3$, with $S^{(2)}$ acting on the first two factors.

We therefore have an isomorphism of $H \otimes S^{(2)}$-dimodules

$$f = \tau_1 \circ \beta : P_1 \otimes_2 I = I_1 \otimes_3 I_3 \rightarrow I_2 \rightarrow P_2.$$

Note that τ_1 and β are no $H \otimes S^{(2)}$-dimodule isomorphisms; their composition, however, preserves the $H \otimes S^{(2)}$-dimodule structure. Now consider the map

$$f_2^{-1} f_3 f_1 : P_{11} \otimes_3 I_1 \otimes_3 I_3 \to P_{11} \otimes_3 I_2.$$

Observe that

$$f_1 = \tau_{12}\beta_1,$$
$$f_3 = \tau_{14}\beta_4 = \tau_{14}\tau_{12}\beta_3\tau_{12},$$
$$f_2 = \tau_{14}\tau_{12}\beta_2.$$

Therefore

$$f_2^{-1} f_3 f_1 = \beta_2^{-1}\tau_{12}\tau_{14}\tau_{14}\tau_{12}\beta_3\tau_{12}\tau_{12}\beta_1 = \beta_2^{-1}\beta_3\beta_1.$$

Now $\beta_2^{-1}\beta_3\beta_1$, or more precisely $\beta_2^{-1}\beta_3(\beta_1 \otimes I_{33}) = I_{11} \circ \beta_4$, because $\delta_1\alpha = \lambda_1$. This means that

$$f_2^{-1} f_3 f_1 : P_{11} \otimes_3 I_1 \otimes_3 I_3 \to P_{11} \otimes_3 I_2$$

induces the identity of $\mathrm{End}_3(P_{11})$. Therefore

$$\mathrm{End}(f) : \mathrm{End}_2(P_1) \to \mathrm{End}_2(P_2)$$

is a descent datum. So $\mathrm{End}_S(P)$ descends to an H-dimodule algebra A, which is an H-Azumaya algebra. It is not difficult to show that $i([A]) = [(I, \alpha)]$.

3.9. *Theorem*

Let R be a commutative ring, and H a faithfully projective, commutative and cocommutative Hopf algebra. Then

$$\mathrm{BD}^s(R, H) \cong (H^1(R, G(H^* \otimes .))) \times H^1(R, G(H^* \otimes .)) \times_\varphi H^2(R, \mathbb{G}_m)_{\mathrm{tors}}$$
$$\cong (\mathrm{Gal}^s(R, H) \times \mathrm{Gal}^s(R, H^*)) \times_\psi \mathrm{Br}(R).$$

Proof. This follows after taking inductive limits in 3.7 and 3.8 over all faithfully flat extensions S of R. One also has to invoke Corollary 2.8, Theorem 3.4, and Gabber's Theorem (cf. [19]):

$$\mathrm{Br}(R) \cong H^2(R_{\text{ét}}, \mathbb{G}_m)_{\mathrm{tors}}$$

and the formula

$$H^2(R_{\mathrm{fl}}, \mathbb{G}_m)_{\mathrm{tors}} \cong H^2(R_{\text{ét}}, \mathbb{G}_m)_{\mathrm{tors}}$$

(cf. [26, III.3.9]).

4. The Brauer–Long group of a Hopf algebra of rank two

The methods developed above permit us to compute some explicit examples of $BD^s(R, H)$, in the case where H is not a group ring or the dual of a group ring. Let us consider the easiest possible example; let R be a commutative domain of characteristic different from 2, and fix $a, b \in R$ such that $ab = 2$. Let H_a be the Hopf algebra defined by

$$H_a = R[x]/(x^2 - ax),$$

$$\Delta x = x \otimes 1 + 1 \otimes x - bx \otimes x,$$

$$\varepsilon(x) = 0.$$

This Hopf algebra and its Galois objects have been studied by Tate and Oort ([33]), Kreimer ([22]), and Childs ([14]). The characteristic two case was considered by Nakajima ([29]). Hurley ([20]) has studied similar Hopf algebras of rank p. It may be shown that

$$H_a^* \cong H_b = R[y]/(y^2 - by)$$

where

$$b(y) = -1.$$

Also, it is well-known that $H_2 \cong RC_2$ (the isomorphism is given by $1 - x \to \sigma$). If 2 is invertible in R, then all H_a's are isomorphic. For this reason, we will assume in the sequel that 2 is not invertible in R; if 2 is invertible, then we refer to [16] for a complete treatment of $BD(R, C_2)$.

In Section 2, we have seen that the commutative H_a-Galois objects with normal basis are classified by $H^2_{\text{symm}}(H_a, R, \mathbb{G}_m)$. It may be shown that all H_a-Galois objects are commutative, and that

$$H^2_{\text{symm}}(H_a, R, \mathbb{G}_m) \cong U_{b^2}(R)/(U_b(R))^2,$$

where $U_b(R) = \{u \in \mathbb{G}_m(R) : a \equiv 1 \bmod bR\}$. Let us briefly sketch this isomorphism, leaving details to the reader. $H^2(H_a, R, \mathbb{G}_m) = \mathrm{Ker}(\Delta_2)/\mathrm{Im}(\Delta_1)$, where

$$\mathbb{G}_m(H^*) \xrightarrow{\Delta_1} \mathbb{G}_m(H^{*(2)}) \xrightarrow{\Delta_2} \mathbb{G}_m(H^{*(3)}).$$

Let

$$\mathbb{G}_m^n(H^{*(2)}) = \{ f \in \mathbb{G}_m(H^{*(2)}) : f(1 \otimes h) = f(h \otimes 1) = \varepsilon(h), \text{ for all } h \in H \},$$

$$\mathbb{G}_m^n(H^*) = \{ f \in \mathbb{G}_m(H^*) : f(1) = 1 \},$$

and restrict the above sequence to

$$\mathbb{G}_m^n(H^*) \xrightarrow{\Delta_1^n} \mathbb{G}_m^n(H^{*(2)}) \xrightarrow{\Delta_2^n} \mathbb{G}_m(H^{*(2)}).$$

Then

$$H^2(H_a, R, \mathbb{G}_m) = \mathrm{Ker}(\Delta_2^n)/\mathrm{Im}(\Delta_1^n).$$

So we may restrict our attention to looking for "normal" cocycles. It may be shown that every $f \in \mathbb{G}_m^n(H^{*(2)})$ is a cocycle. Furthermore, we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{G}_m^n(H^*) & \xrightarrow{\Delta_1} & \mathbb{G}_m^n(H^{*(2)}) \\ \downarrow \theta_1 & & \downarrow \theta_2 \\ U_b(R) & \xrightarrow[\text{square}]{} & U_{b^2}(R) \end{array}$$

where θ_1, θ_2 are isomorphisms defined by $\theta_1(f) = 1 - bf(x)$ and $\theta_2(f) = 1 + b^2 f(x \otimes x)$. Take $u \in U_{b^2}(R)$, and let

$$e = \frac{u-1}{b^2}.$$

The H_a-Galois object corresponding to u is given by

$$H_a(e) = R[s]/(s^2 - as - e),$$

$$\chi(s) = s \otimes 1 + 1 \otimes x - bs \otimes x.$$

$H_a^* \cong H_b$ acts on $H_a(e)$ as follows:

$$y \to s = bs - 1.$$

Consider $v \in U_{a^2}(R)$, and let

$$f = \frac{v-1}{a^2}.$$

Then we have an H_b-Galois object:

$$H_b(f) = R[t]/(t^2 - bt - f),$$

$$\chi(t) = t \otimes 1 + 1 \otimes y - at \otimes y.$$

Let us compute the smash product $H_a(e) \# H_b(f) = R^{(u,v)}$. Write $\alpha = s \# 1$, $\beta = 1 \# t$, then

(4.1.1) $\alpha^2 - a\alpha - e = 0$,

(4.1.2) $\beta^2 - b\beta - f = 0$,

and

$$\begin{aligned}\beta\alpha &= (1 \# t)(s \# 1)\\ &= \sum_{(t)} (t_{(1)} \rightharpoonup s) \# t_{(0)}\\ &= (s \# t) + ((bs - 1) \# 1) - a((bs - 1) \# t)\\ &= -s \# t + bs \# 1 - 1 \# 1 + a(1 \# t)\\ &= -\alpha\beta + b\alpha + a\beta - 1,\end{aligned}$$

so

(4.1.3) $\beta\alpha + \alpha\beta + 1 = b\alpha + a\beta$.

(4.1.1–3) determine the multiplication in the smash product $R^{(u,v)}$; these rules may be rewritten as follows: let A_1 be the quaternion algebra ${}^uR^v$, that is A_1 is generated by i, j, $ij = k$, where $i^2 = u$, $j^2 = v$, $ij = -ji$. Let $R^{(u,v)}$ be the algebra generated by A_1 and

$$\alpha = \frac{1+i}{b} \quad \text{and} \quad \beta = \frac{1+j}{a}.$$

It may be verified easily that α and β satisfy (4.1.1–3).

The above computations and Theorem 3.9 imply:

4.1. *Proposition*

Let R be a domain of characteristic different from 2, *and consider $a, b \in R$ such that $ab = 2$. If all H_a- and H_b-Galois objects have normal basis, then*

$$\mathrm{BD}^s(R, H_a) = U_{b^2}(R)/(U_b(R))^2 \times U_{a^2}(R)/(U_a(R))^2 \times \mathrm{Br}(R),$$

with multiplication rules

$$(u, v, [A])(u', v', [A']) = (uu', vv', [A \otimes B \otimes R^{(u,v')}]).$$

4.2. *Example*

Let us give a more specific example. In 4.1, let $R = \mathbb{Z}[\sqrt{2}]$, and $a = b = \sqrt{2}$. Then $H_{\sqrt{2}}$ is a self-dual Hopf algebra, and $U_2(R) = \{-1, 1\}$, and $U_{\sqrt{2}}(R) = \{1\}$. Hence

$$\mathrm{Gal}(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) \cong \mathbb{Z}/2\mathbb{Z}$$

(all Galois objects are commutative and have normal basis in this case), and the only nontrivial Galois object is

$$H_{\sqrt{2}}(-1) = \mathbb{Z}\left[\sqrt{2}, \frac{1+i}{\sqrt{2}}\right];$$

$A = H_{\sqrt{2}}(-1) \# H\sqrt{2}(-1)$ is the sub-$\mathbb{Z}[\sqrt{2}]$-algebra of $\mathbb{H}$ generated by i, j, $(1+i)/\sqrt{2}$ and $(1+j)/\sqrt{2}$. Actually, A represents the only nontrivial element of $\mathrm{Br}(\mathbb{Z}[\sqrt{2}])$, and is the most elementary example of an Azumaya algebra which is not equivalent to a classical Galois crossed product (cf. [21, IV.6.5]). It follows from 4.1 that $\mathrm{BD}^s(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}})$ is a group containing 8 elements. Examining the multiplication rules in 4.1, we obtain

$$\mathrm{BD}^s(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) \cong D_4,$$

the dihedral group of 8 elements.

What about the complete Brauer–Long group $\mathrm{BD}(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}})$? Using the fact that $H_{\sqrt{2}}$ is self-dual, we may construct a noncentral $H_{\sqrt{2}}$-Azumaya algebra. Let

$$A = H_{\sqrt{2}} = \mathbb{Z}[\sqrt{2}]/(x^2 - \sqrt{2}x)$$

as a $\mathbb{Z}[\sqrt{2}]$-algebra, with an $H_{\sqrt{2}}$-dimodule structure given by

$$x \rightharpoonup 1 = 0; \quad x \rightharpoonup x = \sqrt{2}x - 1,$$

$$\chi(x) = \Delta x = x \otimes 1 + 1 \otimes x - \sqrt{2}x \otimes x.$$

4.3. *Proposition*

A is an $H_{\sqrt{2}}$-Azumaya algebra.

Proof. We have to show that $F: A \# \bar{A} \to \mathrm{End}_R(A)$ and $G: \bar{A} \# A \to \mathrm{End}_R(A)^{\mathrm{opp}}$ are surjective maps. We only show that F is surjective, the other part is left to the reader. Fix $\{1, x\}$ as a basis for $H_{\sqrt{2}}$. Then $\mathrm{End}(A) = M_2(\mathbb{Z}[\sqrt{2}])$. A straightforward computation yields

$$F(1\#1)=\begin{pmatrix}1&0\\0&1\end{pmatrix};\qquad F(x\#1)=\begin{pmatrix}0&0\\1&\sqrt{2}\end{pmatrix},$$

$$F(1\#x)=\begin{pmatrix}0&-1\\1&\sqrt{2}\end{pmatrix};\quad F(x\#x)=\begin{pmatrix}0&0\\\sqrt{2}&1\end{pmatrix}.$$

We leave it to the reader to check that these four matrices generate $M_2(\mathbb{Z}[\sqrt{2}])$.

4.4. *Conjecture*

$$\mathrm{BD}(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}})/\mathrm{BD}^s(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) = C_2,$$

and consequently

$$\mathrm{BD}(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) = D_8.$$

We provide an indication how to prove this conjecture. One may generalize the classical property that the Brauer group of a regular domain maps injectively into the Brauer group of its Brauer group; that is, proceeding as in [32, 6.19], one should be able to show that

$$\mathrm{BD}(R, H) \to \mathrm{BD}(K, K\otimes H)$$

is monomorphic if R is a regular domain. Now let $S = \mathbb{Z}_2[\sqrt{2}]$; then $S\otimes H_{\sqrt{2}} = SC_2$, and the above statement then implies that

$$\mathrm{BD}(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) \to \mathrm{BD}(S, SC_2)$$

is monomorphic; hence

$$\mathrm{BD}(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}})/\mathrm{BD}^s(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) \to \mathrm{BD}(S, SC_2)/\mathrm{BD}^s(S, SC_2) = C_2$$

is monomorphic, and the statement follows. To see that this implies that

$$\mathrm{BD}(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) = D_8,$$

observe that we have a homomorphism

$$\mathrm{BD}(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) \to \mathrm{BD}(\mathbb{R}, \mathbb{R}C_2).$$

Counting elements, we see that this homomorphism is an isomorphism; it was shown by Long that $\mathrm{BD}(\mathbb{R}, \mathbb{R}C_2) = D_8$.

Of course it would be useful to know $\mathrm{BD}(R, H)/\mathrm{BD}^s(R, H)$; this would, for instance, solve the conjecture above, without one having to adapt [32, 6.19]. For $H = RG$, where G is an abelian group such that $|G|$ is invertible in R, and such that G^{**} and G are naturally isomorphic, the quotient $\mathrm{BD}(R, H)/\mathrm{BD}^s(R, H)$ was computed by the author and M. Beattie (cf. [10, Sec. 3]); using some generalizations of the Skolem–Noether Theorem, it was shown that $\mathrm{BD}(R, RG)/\mathrm{BD}^s(R, RG)$ is a well-defined subgroup of $\mathrm{Aut}(G \times G^*)$. I believe that this property may be generalized to the Hopf-algebra situation, and that $\mathrm{BD}(R, H)/\mathrm{BD}^s(R, H)$ is a specific subgroup of $\mathrm{Aut}_{\mathrm{Hopf}}(H \times H^*)$. This will actually be the topic of [9]; it will allow us to describe the full Brauer-Long group as the middle term of a short exact sequence. Returning to the example of a rank two Hopf algebra, it will allow us to solve the following

4.5. *Conjecture*

Let R be a domain of characteristic different from 2, and suppose that 2 is not invertible in R. Let $a, b \in R$ such that $ab = 2$. Then

$$\mathrm{BD}(R, H_a) = \mathrm{BD}^s(R, H_a) \qquad \text{if } a \neq \sqrt{2},$$

$$\mathrm{BD}(R, H_a)/\mathrm{BD}^s(R, H_a) = C_2 \qquad \text{if } a = \sqrt{2}.$$

5. Orzech's subgroup of the Brauer–Long group

As before, let H be a faithfully projective, commutative and cocommutative Hopf algebra. Fix a Hopf algebra homomorphism $\theta : H \to H^*$, and consider an H-dimodule M with structure maps χ and ψ. Recall that the H-module structure on M defines an H^*-comodule structure on M; the H^*-comodule structure map thus defined will be called φ.

5.1. *Definition* (Orzech [31])

M is called a θ-module if and only if

$$\varphi = (1 \otimes \theta)\chi.$$

An H-dimodule algebra which is a θ-module is called a θ-algebra.

Observe that the H-action on a θ-module M is given by

$$h \rightharpoonup m = \sum_{(m)} \theta(m_{(1)})(h) m_{(0)}.$$

As announced by Orzech, it is straightforward to check that, for two θ-modules M, N, $M \otimes N$ and $\mathrm{Hom}_R(M, N)$ are again θ-modules. The H-

opposite of a θ-algebra, and the smash product of two θ-algebras are again θ-algebras. Therefore we have

5.2. *Proposition* (Orzech [31])

The set $B_\theta(R, H) = \{\alpha \in \mathrm{BD}(R, H) : \alpha$ *is represented by a* θ*-algebra*$\}$ *is a subgroup of* $\mathrm{BD}(R, H)$.

$B_\theta(R, H)$ is a generalization of the Brauer–Wall group: if 2 is invertible in R, and if θ is the unique Hopf algebra isomorphism $RC_2 \to (RC_2)^*$, then $B_\theta(R, H) = \mathrm{BW}(R)$. In a similar way, the Brauer group introduced by Childs, Garfinkel and Orzech is a special case of $B_\theta(R, H)$. If $\theta : H \to H^*$ is the trivial map, then $B_\theta(R, H) = \mathrm{BC}(R, H)$. Moreover, the Brauer–Long group itself is a special case of $B_\theta(R, H)$. Indeed, we have the following generalization of [13, 5.1], solving a question raised by Orzech ([31, Sec. 3]):

5.3. *Proposition*

Let $\theta : H \otimes H^* \to (H \otimes H^*)^* = H^* \otimes H$ *be defined by*

$$\theta(h \otimes h^*) = h^* \otimes \eta\varepsilon(h).$$

Then $\mathrm{BD}(R, H) = B_\theta(R, H \otimes H^*)$.

Proof. Let θ-mod be the full subcategory of $H \otimes H^*$-dim, consisting of θ-modules. Forgetting the $H \otimes H^*$-module structure, we obtain a functor

$$\theta\text{-mod} \to \text{com-}H \otimes H^*.$$

It is clear that this functor is an equivalence of categories. Then, applying the category equivalence between com-$H \otimes H^*$ and H-dim (cf. 1.1), we see that θ-mod and H-dim are equivalent categories. After identifying the involved categories, our result follows. The only problem left is the following: given two θ-module algebras A and B, do the smash products $A \# B$ (with respect to the H-dimodule structure of A and B) and $A \#_\theta B$ (with respect to the $H \otimes H^*$-dimodule structure of A and B defined by θ) coincide?

Write $J = H \otimes H^*$, and write χ_J for the J-comodule structure map. For $a \in A$, write

$$\chi_J(a) = \sum_{(a)} a_{(0)} \otimes a_{(1)} \otimes a^*_{(1)} \in A \otimes H \otimes H^*.$$

Then the H-comodule structure map, and the H^*-comodule structure map on A are given by

$$\chi_H(a)=\sum_{(a)}\varepsilon(a^*_{(1)})a_{(0)}\otimes a_{(1)}\in A\otimes H,$$

$$\chi_{H^*}(a)=\sum_{(a)}\varepsilon(a_{(1)})a_{(0)}\otimes a^*_{(1)}\in A\otimes H^*.$$

The H-action on A is therefore given by

$$h\rightarrow a=\sum_{(a)}\varepsilon(a_{(1)})a^*_{(1)}(h)a_{(0)}.$$

In $A\#_\theta B$, we have

$$\begin{aligned}(a\#b)(c\#d)&=\sum_{(b)}a((b_{(1)}\otimes b^*_{(1)})\rightarrow c)\#b_{(0)}d\\ &=\sum_{(b),(c)}a\theta(c_{(1)}\otimes c^*_{(1)})((b_{(1)}\otimes b^*_{(1)}))c_{(0)}\#b_{(0)}d\\ &=\sum_{(b),(c)}ac^*_{(1)}(b_{(1)})b^*_{(1)}(\varepsilon\eta c_{(1)})c_{(0)}\#b_{(0)}d.\end{aligned}$$

In $A\#B$, we have

$$\begin{aligned}(a\#b)(c\#d)&=\sum_{(b)}a(b_{(1)}\rightarrow c)\#\varepsilon^*(b^*_{(1)})b_{(0)}d\\ &=\sum_{(b),(c)}a\varepsilon^*(b^*_{(1)})\varepsilon(c_{(1)})c^*_{(1)}(b_{(1)})c_{(0)}\#b_{(0)}d.\end{aligned}$$

Therefore $A\#_\theta B=A\#B$, since

$$\varepsilon^*(b^*_{(1)})\varepsilon(c_{(1)})=b^*_{(1)}(\eta(1))\varepsilon(c_{(1)})=b^*_{(1)}(\eta\varepsilon c_{(1)}).$$

As an application of the results of Section 3, we compute $B^s_\theta(R,H)$. Observe that $\theta:H\rightarrow H^*$ induces maps $\theta_1=\mathrm{Gal}(R,\theta)$ and $\theta_2=H^1(R,G(\theta))$ such that we have a commutative diagram

$$\begin{array}{ccc}\mathrm{Gal}^s(R,H)&\cong&H^1(R,G(H^*\otimes .))\\ \downarrow\theta_1&&\downarrow\theta_2\\ \mathrm{Gal}^s(R,H^*)&\cong&H^1(R,G(H\otimes .))\end{array}$$

5.4. *Proposition*

Let $\theta : H \to H^*$ be a homomorphism of Hopf algebras. Then

$$B^s_\theta(R, H) \cong H^1(R, G(H^* \otimes .)) \times H^2(R, \mathbb{G}_m)_{\text{tors}}$$

$$\cong \mathrm{Gal}^s(R, H) \times \mathrm{Br}(R).$$

The multiplication rules are given by

$$(h_1^*, u_1)(h_2^*, u_2) = (h_1^* h_2^*, u_1 u_2 \varphi(h_1^*, \theta^* h_2^*))$$

on the cocycle level, and by

$$([S], [A])([T], [B]) = ([S][T], [A \otimes B \otimes (S \# \theta_1(T))])$$

on the algebra level. The embedding in the Brauer–Long group is given by

$$(h^*, u) \to (h^*, \theta^*(h^*), u),$$

$$(S, A) \to (S, \theta_1(S), A).$$

PROOF. This follows directly from the observations made above and from Theorem 3.9.

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